A Simpler Proof of the Characterization of Quadric CMC Hypersurfaces in $S^{n+1}$

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Abstract

In this short article, we present a new and simpler proof of a characterization of the quadric constant mean curvature hypersurfaces of the Euclidean sphere $S^{n+1}$, originally due to Alias, Brasil and Perdomo

Keywords

Euclidean sphere; Constant mean curvature hypersurfaces; Support functions; Totally umbilical hypersurfaces; Clifford torus

Introduction

In 2008, Alias, Brasil and Perdomo studied complete hypersurfaces immersed in the unit Euclidean sphere $S^{n+1} \subset \mathbb{R}^{n+2}$, whose height and angle functions with respect to a fixed nonzero vector of the Euclidean space $\mathbb{R}^{n+2}$ are linearly related. Let us recall that, for a fixed arbitrary vector $\mathbf{a} \in \mathbb{R}^{n+2}$ the height and the angle functions naturally attached to a hypersurface $\varphi : \Sigma^n \to \mathbb{R}^{n+2}$ endowed with an orientation $\nu$ are defined, respectively, by $l_\varphi = \langle \varphi, \mathbf{a} \rangle$ and $f_\varphi = \langle \nu, \mathbf{a} \rangle$. In this setting, they showed the following characterization result concerning the quadric constant mean curvature hypersurfaces of $S^{n+1}$ [1,2]:

**Theorem 1**

Let $\varphi : \Sigma^n \to S^{n+1} \subset \mathbb{R}^{n+2}$ be a complete hypersurface immersed in $S^{n+1}$ with constant mean curvature $l_\varphi = l_0$ for some non-zero vector $\mathbf{a} \in \mathbb{R}^{n+2}$ and some real number $\lambda$, then $\Sigma^n$ is either a totally umbilical hypersurface or a Clifford torus $S^k(\rho) \times S^{n-k}(\sqrt{1-\rho^2})$, for some $k = 0; 1; \ldots; n$ and some $\rho > 0$.

Later on, working with a different approach of that used in [2], the first and second authors characterized the totally umbilical and the hyperbolic cylinders of the hyperbolic space $H^n$ as the only complete hypersurfaces with constant mean curvature and whose support functions with respect to a fixed nonzero vector $\mathbf{a}$ of the Lorentz-Minkowski space are linearly related (see Theorem 4:1 of [3,4], for the case that $\mathbf{a}$ is either space like or time like, and Theorem 4:2 of [5], for the case that $\mathbf{a}$ is a nonzero null vector). In this short article, our purpose is just to use a similar approach of that in [4,5] in order to present a new and more simple proof of Theorem 1 (cf. Section 3). For this, in Section 2 we recall some preliminaries facts concerning hypersurfaces immersed in $S^{n+1}$.

Preliminaries

Let $\varphi : \Sigma^n \to S^{n+1} \subset \mathbb{R}^{n+2}$ be an orientable hypersurface immersed in the Euclidean sphere. We will denote by $\mathbf{A}$ the Weingarten operator of $\Sigma^n$ with respect to a globally defined unit normal vector $\nu$.

In order to set up the notation, let us represent by $\nabla$, $\mathcal{A}$ and $\mathcal{V}$ the Levi-Civita connections of $\mathbb{R}^{n+2}$, $S^{n+1}$ and $\Sigma^n$ respectively. Then the Gauss and Weingarten formulas for $\Sigma^n$ in $S^{n+1}$ are given, respectively, by

$$
\nabla_\varphi Y = \nabla_Y \varphi + \langle \mathcal{A}X, Y \rangle \nu - \langle X, Y \rangle \varphi
$$

and

$$
\mathcal{A}X = -\nabla_\mathcal{A}X = -\nabla_X \mathcal{A}X,
$$

for all tangent vector fields $X,Y$ on $\varphi : \Sigma^n \to S^{n+1}$.

In what follows, we will work with the three symmetric elementary functions of the principal curvatures $\lambda_1, \ldots, \lambda_n$ of $\varphi$, namely:

$$
S_1 = \sum \lambda_i, \quad S_2 = \sum \lambda_i \lambda_j, \quad S_3 = \sum \lambda_i \lambda_j \lambda_k
$$

where $i, j, k \in \{1, \ldots, n\}$.

As before, for a fixed arbitrary vector $\mathbf{a} \in \mathbb{R}^{n+2}$ let us consider the height and the angle functions naturally attached to $\varphi$ which are defined, respectively, by $l_\varphi = \langle \varphi, \mathbf{a} \rangle$ and $f_\varphi = \langle \nu, \mathbf{a} \rangle$. A direct computation allows us to conclude that the gradient of such functions are given by $\nabla l_\varphi = a^1$ and $\nabla f_\varphi = -a^1$, where $a^1$ is the orthogonal projection of $\mathbf{a}$ onto the tangent bundle $T\Sigma^n$ that is,

$$
a^1 = a - f_\varphi \nu - l_\varphi \mathbf{A}.
$$

Taking into account that $\nabla a^l = 0$ and using Gauss and Weingarten formulas, we obtain $\nabla l_\varphi = f_\varphi \mathcal{A}X + l_\varphi X$ for all $X \in T\varphi(M)$. We use this previous identity jointly with Codazzi equation to deduce that

$$
\nabla_X a^1 = f_\varphi \mathcal{A}X + l_\varphi X + (\nabla_\varphi a^1)(X),
$$

for all $X \in T\varphi(M)$. Thus according to [6] (see also [3]), it follows from the last two identities that

$$
\begin{align*}
\nabla l_\varphi &= nh f_\varphi - n l_\varphi \quad (2.1) \\
\nabla f_\varphi &= -|a|^2 f_\varphi + nh f_\varphi - n \langle \nabla H, a^1 \rangle \\
\end{align*}
$$

where $H = (1/n) S_1$ is the mean curvature function of $\varphi$.

For what follows, it is convenient to consider the so-called Newton transformation

$$
P : \mathcal{A}(\Sigma) \to H(\Sigma)
$$

$$
P = S_1^{-1} \mathcal{A}
$$

where $I$ is the identity operator. Naturally associated with the Newton transformation $P$, we have the Cheng-Yau’s square operator $[7]$, which is the second order linear differential operator $\mathcal{D}(\Sigma) \to D(\Sigma)$ given by

$$
\mathcal{D} \mathcal{D} = tr(P \circ \nabla^2 h)
$$

(2.4)
Here $\nabla^2 h : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma)$ stands for the self-adjoint linear operator metrically equivalent to the hessian of $h$, and it is given by

$$\left\{ \nabla^2 h(X,Y) \right\} = \{ (\nabla_{\nabla_X Y} h) \}_{X \in \mathcal{A}(\Sigma), Y \in \mathcal{A}(\Sigma)}$$

For all $X, Y \in \mathcal{A}(\Sigma)$.

Based on Reilly’s seminal paper [8-10], Rosenberg [6] showed the following identifications related to the action of $\square$ on the functions $f_i$ and $f_j$:

$$\Box f_i = 2s_i f_i - (n-1)s_i f_i \tag{2.5}$$

And

$$\Box f_i = (s_i s_j - 3s_i) f_i + 2s_i f_i - \langle \nabla s_i, a_i^2 \rangle \tag{2.6}$$

To close this section, we quote a suitable Simons-type formula which can be found in [1] [11].

$$\Box s_i = \Delta s_i + \sum_{j=0}^{n} (s_j s_i - 3s_i) s_j + 3s_i s_j + (n-1)s_i^2 \tag{2.7}$$

**Proof of Theorem**

Now, we are in position to proceed with our alternative proof of Theorem 1.1. If $\lambda = 0$ then $l = 0$ and

$$\psi(x, a) = \frac{1}{|a|} \psi(x, a) = 0$$

for all $x \in \Sigma^*$ and, consequently, $\Sigma^*$ is a totally umbilical sphere of $S^{n+1}$.

So, let us assume that $\lambda \neq 0$. We have $\Delta l = \Delta f_i$ and using the fact that $H$ is constant, from (2.1) and (2.2) we conclude that

$$nH f_i = - \lambda \langle A \rangle f_i + \lambda nH f_i$$

Or equivalently,

$$S_i f_i = - \lambda S_i f_i - 2s_i f_i + \lambda S_i f_i = - \lambda S_i f_i + 2\lambda S_i f_i + \lambda S_i f_i$$

Hence, we get that

$$S_i f_i = - \lambda S_i f_i + 2\lambda S_i f_i + \lambda S_i f_i = 0 \tag{3.1}$$

By (3.1), we obtain

$$0 = \lambda (S_i f_i - n f_i + \lambda S_i^2 f_i - 2\lambda S_i f_i - \lambda S_i f_i)$$

$$S_i f_i - n f_i + \lambda S_i^2 f_i - 2\lambda S_i f_i - \lambda S_i f_i$$

$$S_i f_i - n f_i + \lambda S_i^2 f_i - 2\lambda S_i f_i$$

Thus,

$$(S_i - n \lambda + \lambda S_i^2 - 2\lambda S_i - \lambda^2 S_i) f_i = 0 \tag{3.2}$$

We define a function

$$h : \Sigma^* \to \mathbb{R}$$

$$h(p) = (S_i - n \lambda + \lambda S_i^2 - 2\lambda S_i - \lambda^2 S_i) f_i(p)$$

Suppose that $h(p) \neq 0$ for some $p \in \Sigma^*$. Since $h$ is smooth, there exists a neighbourhood $u$ of $p$ in $\Sigma^*$ in which $h(p) \neq 0$ for all $p \in u$. From (3.2) we conclude that $l = 0$ in $u$ and, hence $f_i = 0$ in $u$, since $\lambda \neq 0$. We arrive at a contradiction because in $\Sigma^*$ we have

$$|\nabla f_i|^2 + f_i^2 = |a|^2 \geq 0$$

Therefore, $h = 0$ on $\Sigma^*$, that is,

$$S_i - n \lambda + \lambda S_i^2 - 2\lambda S_i - \lambda^2 S_i = 0 \tag{3.3}$$

Consequently, $S_i$ is constant on $\Sigma^*$. Repeating the previous argument for the operator $L$, and using the fact that $S_i$ is constant, we also obtain that

$$2S_i - \lambda (n-1)S_i + \lambda S_i - 3\lambda S_i - 2\lambda S_i = 0 \tag{3.4}$$

We observe that the above equation shows that $S_i$ is also constant on $\Sigma^*$. We also note that this argument shows, in fact, that $S_i$ is a constant function on $\Sigma^*$ for all $2s_i \leq n$. From (2.7) we get

$$|\nabla A|^2 + 2S_i (S_i - 2S_i - n) - S_i (S_i - 3S_i - (n-1)S_i) = 0 \tag{3.10}$$

More precisely,

$$|\nabla A|^2 + 2S_i (S_i - 2S_i - n) - S_i (S_i - 3S_i - (n-1)S_i) = 0 \tag{3.5}$$

We observe that if $H = 0$, then $S_i = 0$ and, consequently, $|A|^2 = 2s_i$. From (3.3), we have $2S_i = -n$ and $|A|^2 = n$. Therefore, since

$$\frac{1}{2} \Delta |A|^2 = n |A|^2 - |A|^4 - |
abla |A|^2|$$

We have that $|\nabla |A|^2| = 0$ and, hence, from Theorem 4 of [9], we conclude that $\Sigma^*$ must be a Clifford torus.

$$S^i(p) \times S^{n-1}(\sqrt{1-p^2})$$

for some $k=0,1,\ldots,n$ and some $p>0$.

Now, suppose that $H \neq 0$. By equation (3.4) we get

$$2S_i S_i - \lambda(n-1)S_i + \lambda S_i S_i - 2\lambda S_i S_i = 0 \tag{3.6}$$

that is,

$$2S_i S_i - \lambda(n-1)S_i + \lambda S_i S_i - 2\lambda S_i S_i = 0 \tag{3.7}$$

From equation (3.5) we have

$$\lambda |\nabla A|^2 + \lambda S_i S_i - 4\lambda S_i S_i - 2n S_i S_i + 3\lambda S_i S_i + \lambda(n-1)S_i = 0 \tag{3.8}$$

Furthermore, from a straightforward computation we can verify that

$$\lambda |\nabla A|^2 + \lambda S_i S_i - 4\lambda S_i S_i - 2n S_i S_i + 3\lambda S_i S_i + \lambda(n-1)S_i = 0 \tag{3.9}$$

Hence, if $S_i = 0$ we obtain of (3.9) that $\lambda |\nabla A|^2 = 0$ consequently, $|\nabla |A|^2| = 0$ and, since $\Sigma^*$ is complete, it follows once more from Theorem 4 of [9] that $\Sigma^*$ must be a Clifford torus.

If $S_i = 0$ then $2S_i (S_i - n \lambda + \lambda S_i S_i - 2\lambda S_i - \lambda^2 S_i) = 0$ implies

$$2S_i = 2n S_i S_i + 2\lambda S_i S_i - 4\lambda S_i S_i - 2\lambda S_i S_i = 0 \tag{3.10}$$

We note that (3.10) and (3.9) imply $\lambda |\nabla A|^2 = 0$ and, hence, repeating the previous argument we also get that $\Sigma^*$ is a Clifford torus. Therefore, we conclude the proof of Theorem 1.

**References**


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