From Configurations to Branched Configurations and Beyond
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Abstract
We propose here a geometric and topological setting for the study of branching effects arising in various fields of research, e.g. in statistical mechanics and turbulence theory. We describe various aspects that appear key points to us, and finish with a limit of such a construction which stand in the dynamics on probability spaces where it seems that branching effects can be fully studied without any adaptation of the framework.

Keywords
Finite and infinite configuration spaces; Branched dynamics

Introduction
Finite and infinite configuration spaces are rather old topics, see e.g. [1,2] and the references cited therein, that had many applications in various settings in mathematical physics and representation theory. More recently, several papers, including [3-6] showed how these topics could be applied in various disciplines: ecology, financial markets, and so on. This large spectrum of applications principally comes from the simplicity of the model: considering a state space $N$, finite or infinite configurations are finite or countable sets of values in $N$. This is why we begin with giving a short description of this setting, and describe a differentiable structure that can fit with easy problems of dynamics. This structure, which can be seen either as a Frölicher structure or as a diffeological one, is carefully described and the links between these two frameworks are summarized in the appendix. We also give a result that seems forgotten in the past literature: the structure or as a diffeological one, is carefully described and the links between these two frameworks are summarized in the appendix.

But the main goal of this paper is to include one dimensional turbulence effects (in particular period doubling, see e.g. [8]) in the dynamics described by finite or infinite configuration spaces. For this, we need to define a symmetric binary relation $u$ on $I$, that expresses the compatibility of two physical quantities. We assume also that $u$ has the following property:

$$\forall (u, v) \in (\Gamma^i)^2, uUv \Rightarrow u \neq v$$

In the settings Albeverio et al. [7] and Finkelstein et al. [1], two particles cannot have the same position. Then, for $x, y \in N$ [2],

$$xUy \Leftrightarrow x \neq y$$

With these restrictions, we can define the indexed or ordered configuration spaces:

$$\text{Of}^i = \{(u_1, ..., u_n) \in (I^i)^n \text{ such that, } i \neq j; u_iu_j \}$$

$$\text{Of} = \{ \text{Of}^i \}$$

The general configuration spaces are not ordered. Let $\Sigma_o$ be the group of bijections on $N_o$, and $\Sigma_i$ be the set of bijections on $I$. We can define two actions:

$$\sum_{i} \times \text{Of}^i \rightarrow \text{Of}^i$$

$$(\sigma, (u_1, ..., u_n)) \rightarrow (u_{\sigma(1)}, ..., u_{\sigma(n)})$$

and its infinite analog:

$$\sum_{i} \times \text{Of}^i \rightarrow \text{Of}^i$$

$$(\sigma, (u_1^i)) \rightarrow (u_{\sigma(1)^i})_{\sigma i}$$

where $\Sigma_i$ is a subgroup of the group of bijections of $I$. In the sequel, $I$ is countable with discrete topology, which avoids topological problems on $\Sigma_o$ as in more complex examples. Then, we define general configuration spaces:

$$\Gamma^i = \text{Of}^i / \Sigma_o$$

in optimal transport, the space of probability measures, and shows how they can also furnish configurations for incompressible fluids. In these settings, branching effects are well-known and sometimes obvious, and we do not need any adaptation of the framework to obtain a full description of them. Therefore, what we call branched configurations appears as an intermediate (an we hope useful) step between dynamics of e.g. a $N$-body problem and e.g. wave dynamics.

Dirac configuration spaces on a locally compact manifold
Let us describe step by step a way to build infinite configurations, as they are built in the mathematical literature. We explain each step with the configurations already defined in e.g. Albeverio et al. [7] and Fadell et al. [1], the generalization will be discussed later in this paper.

A set of $I$-configurations is a set of objects that are modelizations of physical quantities. For example, in the settings [1-6], the physical quantity modeled is the position of one particle. The whole world is modeled as a locally compact manifold $N$, and the set of $1$-configurations is itself $N$, or equivalently the set of all Dirac measures on $N$.

Let $I$ be a set of indexes. $I$ can be countable or uncountable. We define the indexed (or the ordered if $I$ is equipped with its total order) configuration spaces. For this, we need to define a symmetric binary relation $u$ on $I$, that expresses the compatibility of two physical quantities. We assume also that $u$ has the following property:

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\[ \Gamma = \prod_{\alpha \in A} \Gamma^\alpha \]
\[ \Gamma' = O \Gamma' / \Sigma_i \]

Configuration spaces on more general settings

In the machinery of the last section, the properties of the base manifold \( N \) are not used in the definition of the space \( \Gamma' \). This is why the starting point can be \( \Gamma' \) instead of \( N \), and we can give it the most general differentiable structure. Let us first consider the most general case [9]:

**Proposition 1.1:** If \( \Gamma' \) is a diffeological space, then \( \Gamma'' \), \( \Gamma \) and \( \Gamma' \) are diffeological spaces.

**Proof:** \( (\Gamma')^+ \) (resp. \( (\Gamma')^0 \)) is a diffeological space according to Proposition 4.8 (resp. Proposition 4.10), so that, \( O \Gamma^+ \) (resp. \( O \Gamma' \)) is a diffeological space as a subset of \( (\Gamma')^+ \) (resp. \( (\Gamma')^0 \)). Thus, \( \Gamma'' = O / \Sigma_i \) (resp. \( \Gamma' = O^+ / \Sigma \)) has the quotient diffeology by Proposition 4.12, which ends the proof.

Let us now turn to the cases where \( \Gamma' \) has a stronger structure. We already know that \( \Gamma \) is a manifold if \( \Gamma' \) is a manifold.

**Proposition 1.2:** If \( \Gamma' \) is a Frölicher space, then \( \Gamma'' \) (and hence \( \Gamma \)) is a Frölicher space.

**Proof:** Adapting the last proof, using Proposition 4.9 instead of Proposition 4.8, we get that \( O \Gamma^+ \) is a Frolicher space if \( \Gamma' \) is a Frolicher space. Let us now build a generating set of functions for the Frölicher structure on \( O \Gamma'' \). Let \( f : O \Gamma' \to \mathbb{R} \) be a smooth map. We define the symmetrization of \( f \):

\[ f : (u_1, \ldots, u_n) \in O \Gamma'' \mapsto f(u_1^{(1)}, \ldots, u_n^{(1)}) = \frac{1}{n!} \sum_{\sigma \in S_n} f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \]

The set of functions \( \{f\} \) generate the contours on \( O \Gamma'' \), and then, passing to the quotient, generate the contours on \( \Gamma'' \).

Setting a Frölicher structure on indexed configurations is only a straightforward consequence of Proposition 4.10:

**Proposition 1.3:** If \( \Gamma' \) is a Frölicher space, then \( O \Gamma' \) is a Frölicher space.

But the problem of a Frölicher structure on \( \Gamma' \) a little bit more complicated; let us explain why and give step by step the construction of the Frölicher structure. We first notice that the contours of the Frölicher push forward naturally by the quotient map \( O \Gamma^+ \to \Gamma' \). But, if one wants to describe a generating set of functions, by Proposition 4.9, one has to consider all combinations of a finite number of smooth functions \( \Gamma^+ \to \mathbb{R} \). This generating set does not contain any \( \Sigma \)-invariant function, except constant functions. This is why the approach used in the proof of Proposition 1.2 cannot be applied here. For this, one has to consider the set

\[ F_{\Sigma} = \{ f : O \Gamma' \to \mathbb{R} \mid f \text{ is } \Sigma^-\text{-invariant} \} \]

of equivariant functions on \( O \Gamma' \). This discussion can become very quickly naive and we prefer to leave this question to more applied works in order to fit with known examples instead of dealing with too abstract considerations.

**Topological Configuration Spaces**

In this section, we present examples of 1-configurations and their associated configuration spaces. Manifolds will replace the Dirac measures used in Albeverio et al. [7]. In the sequel, \( N \) is a Riemannian smooth locally compact manifold. The 1-configurations considered keep their topological properties, as in the model of elastodynamics (see e.g. Hughes et al. [10]) or in various quantum field theories. Notice also that we do not give compatibility conditions between two 1-configurations: we would like to give the more appropriate conditions in order to fit with the applied models, this is why we leave this point to more specialized works.

**Topological 1-configurations**

We follow here, for example, Hughes T et al. [10].

**Definition 2.1:** Let \( M \) be a smooth compact manifold and \( N \) an arbitrary manifold. We set

\[ \Gamma'(M, N) = C^\infty(M, N) \]

One can also only consider embeddings, and set:

\[ \Gamma'_e(M, N) = Emb(M, N) \]

where \( \text{dim} M < \text{dim} N \), and \( Emb \) is the set of embeddings. The things run as in the first case, since \( Emb(M; N) \subset C^\infty(M; N) \) is an open subset of \( C^\infty(M; N) \):

**Examples of topological configurations**

**Links.** Let \( \Gamma' = Emb(S; N) \). Here, we fix the uncompatibility relation as

\[ y \gamma' \Leftrightarrow \gamma(S') \cap \gamma(S') \neq \emptyset \]

Then, \( \Gamma'_{\text{link}} \) is the space of \( n \)-links of class \( C^0 \), which is a Frechet manifold.

**Triangulations:** Consider the \( n \)-simplex

\[ \Delta_n = \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n \mid \sum t_i = 1 \right\} \]

If \( N \) is a \( n \)-dimensional manifold, a (finite) triangulation \( \sigma \) of \( N \) is such that:

1. \( \sigma \in (\Gamma'_{\Sigma} (\Delta_n, N))^{\#} \)
2. \((T, T') \in \sigma\) such that \( T \equiv T' \) then \( \text{Im}(T) \cap \text{Im}(T') \) is a simplex or a collection of simplexes of each border \( \text{Im}(\partial T) \) and \( \text{Im}(\partial T') \):
3. \( \bigcup_{T \in \sigma} \text{Im} T = N \)

We get by condition 2 a compatibility condition \( \omega \); for which we can build \( O\Gamma'(\Delta, N), \Gamma(\Delta, N) \); \( O\Gamma^\infty(\Delta, N) \) and \( \Gamma^\infty(\Delta, N) \); If \( N \) is compact, the set of triangulations of \( N \) is a subset of \( \Gamma(\Delta, N) \). If \( N \) is non compact and locally compact, the set of triangulations of \( N \) is a subset of \( \Gamma(\Delta, N) \).

More generally, for \( p < \infty \), one can build \( O\Gamma(\Delta, N), \Gamma(\Delta, N) \), \( O\Gamma^\infty(\Delta, N) \) and \( \Gamma^\infty(\Delta, N) \). This example will be discussed in the section 3.

**Strings and membranes:** A string is a smooth surface \( \Sigma \) possibly with boundary, embedded in \( \mathbb{R}^3 \). A membrane is a manifold \( M \) of higher dimension embedded in some \( \mathbb{R}^3 \). We recover here some spaces of the type \( \Gamma'(M) \), which will be also discussed in section 3.

**Branched Configuration Spaces**

**Dirac branched configurations**

As we can see in section 1.1, finite configurations \( \Gamma \) are made of a countable disjoint union. We now fix a metric \( d \) on \( N \): The idea of
branched configurations is to glue together the components \(\Gamma^n\) on the
generalized diagonal, namely, we define the following distance on \(\Gamma\):

**Definition 3.1:** Let \((u, v) \in \Gamma^2\):

\[
d_i(u, v) = \sup_{(x, t) \in \text{maxx}} \{d(x, u), d(x, v)\}
\]

**Proposition 3.2:** \(d_i\) is a metric on \(\Gamma\):

Proof: We remark that \(d_i\) is the Hausdorff distance restricted to \(\Gamma\):

The following proposition traduces the change of topology of \(\Gamma\) into \(B'\) by cut-and-paste property:

**Proposition 3.3:** \(\forall \mathcal{B} \in \mathbb{N}^*\), \(B_i^{n+1} = \Gamma_i^{n+1} \bigcup_{B_i^n}\). \(B_i^n\) where the identification is made along the trace on \(\Gamma_i^{n+1}\) of the \(d_i\) - neighborhoods of \(B_i^n\) in \(\mathbb{N}^i\).

We remark that we can also define a Frolicher structure on \(\Gamma_i\) generated by set functions the set

\[
u \in \bigcup_{\mathcal{B}} \mathbb{R} \rightarrow \frac{1}{|\nu|} \sum_{\nu=\mathcal{B}} f (\nu) | f \in C^\infty (\mathbb{N}, \mathbb{R})\}
\]

This structure will be recovered later in this paper.

**Examples of topological branched configurations**

**The path space, branched paths and graphs:** Let \(\Gamma_i^n((0; 1]; N) = C^\infty((0; 1]; N)\)

be the space of smooth paths on \(N\): A path \(\gamma\) has a natural orientation, and has a beginning \(\alpha(\gamma)\) and an end \(\omega(\gamma)\): We define a compatibility condition

\[
y \gamma \gamma' \Leftrightarrow ((\Im \gamma' - \Im \gamma) \vee (\alpha(\gamma'), \beta(\gamma'))) = (\alpha(\gamma'), \beta(\gamma'))
\]

and we remark that the set of piecewise smooth paths on \(N\) is a subset of \(O_{\text{path}}((0; 1]; N)\), saying that \((\gamma_1, \ldots, \gamma_n) \in O_{\text{path}}((0; 1]; N)\) is a piecewise smooth path if and only if

\[
\forall \gamma_i \in \bigcup_{\mathcal{B}} \mathbb{R} \rightarrow \alpha(\gamma') = (\gamma', + 1)
\]

This relation, stated from the natural definition of the composition * of the groupoid of paths, is not unique and can be generalized.

**Definition 3.4:** Let \((\gamma_1, \ldots, \gamma_n) \in O_{\text{path}}((0; 1]; N)\) and let \(\gamma_n \in C^\infty((0; 1]; N)\): We define the equivalence relation \(\sim\) by

\[
(\gamma_1, \gamma_2, \ldots, \gamma_n) \sim \gamma_1 = \gamma_2 = \ldots = \gamma_n
\]

The maps \(\alpha : \gamma \mapsto \alpha(\gamma')\) and \(\alpha : \gamma \mapsto \alpha(\gamma')\) extends to \(\{\text{set theoretical maps}\} f (\gamma) \in O_{\text{path}}((0; 1]; N)\) and \(\Gamma_i((0; 1]; N) \rightarrow (N)\). The following is now natural:

**Definition 3.5:** A branched path is an element \(\gamma\) of \(O_{\text{path}}(\Gamma_i((0; 1]; N))\) such that, if \(\gamma \in O_{\text{path}}((0; 1]; N)\),

\[
\forall i \in \bigcup_{\mathcal{B}} \mathbb{R} \rightarrow \alpha(\gamma') = (\gamma', + 1) \in \Gamma_i(N)
\]

Example: Let us consider the following paths \([0; 1] \rightarrow \mathbb{R}^2\):

\[
\gamma_1(t) = (t - 2; 0)
\]

\[
\gamma_2(t) = (\cos(\pi(1 - t)); \sin(\pi t))
\]

\[
\gamma_3(t) = (\cos(\pi(1 - t)); -\sin(\pi t))
\]

Then,

\[
\omega(\gamma_1) = \omega(\gamma_2) = \omega(\gamma_3)
\]

And

\[
\omega(\gamma_1) = \omega(\gamma_2) = \omega(\gamma_3)
\]

This shows that

\[
(\gamma_1, \gamma_2, \gamma_3) \in O_{\text{path}}((0; 1] \rightarrow \mathbb{R}^2)
\]

is a branched path of \(\mathbb{R}^2\):

**Alternate approach to branched paths:** branched sections of a fiber bundle.

Let \(\pi : F \rightarrow M\) be a fiber bundle of typical fiber \(F\). Here, \(n \in \mathbb{N}^\infty\). Let \(\pi : F \rightarrow M\) be a fiber bundle over \(M\) with typical fiber \(F\). Let \(\Gamma_i^n(F) = \{u \in \Gamma_i^n(F) : \pi(u) = 1\}

This is trivially a fiber bundle of basis \(M\) with typical fiber \(\Gamma_i^n(F)\):

**Definition 3.6:** A non-section of \(F\) is a section of \(\Gamma_i^n(F)\) which cannot be decomposed into \(n\) sections of \(F\).

We define also \(\Gamma_i^n(F) = \Pi_{\text{path}} \Gamma_i^n(F)\); and also \(\Gamma_i^n(F)\) the non sections based of \(\Gamma_i^n(F)\): We can define the same way \(\Gamma_i^n(F)\) using the branched configuration space instead of the configuration space, since the definitions from the set-theoretic viewpoint are the same.

(Toy) Example: Let us consider the following example: \(X = \mathbb{R}^4(X \{\text{up; down}\}, \Gamma_i^n(X) = \Gamma_i^n(X)\{\{\text{up; down}\}; \{\text{up; down}\}\})\) that models the position of an electron in the 3-dimensional space \(\mathbb{R}^3\), associated to its spin. When the electron spin cannot be determined (i.e. out of the action of adequate electromagnetic fields), the picture proposed by Schrodinger is to consider that its spin is both up and down (this picture is also called the Schrodinger cat "when we replace \{up\} and \{down\} by 'dead' and 'alive').

Let us now consider the Frolicher structure described on section 3.1. It is based on the natural diffeology carried by each \(\Gamma_i^n(F)\) \((\gamma \in \mathbb{N}^\infty)\) and by the set of paths \(P\) that are paths \(\gamma : \mathbb{R} \rightarrow \Gamma_i^n(F)\) such that \(\mathbb{N}(m, n) \in \mathbb{N}^\infty\)

\[
\gamma_{[\mathbb{N}(m, n)]\} \text{ is a smooth path on } \Gamma_i^n(X)
\]

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\]

Let \(\mathbb{R}\gamma(0)\). Then for any smooth map \(f : F \rightarrow \mathbb{R}\)

where \(c^+\) are the trajectories coming from \(l\) in \(0\):

\[
\sum f \circ c^-
\]

Remark that the last condition comes from the smoothness required for each map \(f \circ \gamma\) with \(f \in C^\infty(F) ; R\). This fits with the (fiberwise) Frolicher structure of \(B'\); Then, a finitely branched section of \(F\) is a smooth section of \(B'_i\): The first examples that come to our mind are the well-known branched processes, and we can wonder some deterministic analogues replacing stochastic processes by dynamical systems. Let us here sketch a toy example extracted from the theory of turbulence:

Example: equilibrium of mayies population Assuming that Mayies live and die in the same portion of river, the population \(p_{n+1}\) at the year \(n+1\) is obtained from the population \(p_n\) at the year \(n\) (after
normalization procedure) by the formula
\[ p_{\alpha 1} = A_{\alpha} (1 - p_{\alpha}) \]
where \( A \in [0, 4] \) is a constant coming from the environmental data. For \( A \) small enough, the fixed point of the so-called “logistic map” \( \varphi (x) = Ax (1-x) \) is stable, hence the population \( p_{\alpha} \) tends to stabilize around this value. But when \( A \) is increasing, the fixed point becomes unstable and \( p_{\alpha} \) tends to stabilize around 2\(^n\) multiple values which are the stable fixed points of the map \( \varphi^* \) obtained by composition rule.

Now, assume that we consider a river (or a lake), modeled by an interval (or an open subset of \( \mathbb{R}^2 \)) that we denote by \( U \), where the parameter \( A \) is smooth maps on \( \varphi \). Let \( X \), for any \( \varphi \) of \( X \), and we can state \( \varphi \) is a topological vector space, and hence \( (X, \varphi) \) is a diffeological space.

Let \( \varphi \in X \) be the set \( \varphi \in X \) relation of equivalence by:
\[ R(I, \varphi) \in \varphi \rightarrow \\varphi \rightarrow \varphi \] for any \( \varphi \in X \), and we can state \( \varphi \) is a topological vector space, and hence \( (X, \varphi) \) is a diffeological space.

Non sections in higher dimensions: The example of a lake where mayflies live and die gives us a nice example of branched surface viewed as an element of \( \mathbb{H}^+(\mathbb{R}^3 \times [0;1]) \). The same procedure can be implemented in gluing simplexes, or strings or membranes along their borders to get branched objects, but we prefer to postpone this problem to a work in progress where links with stochastic objects should be performed.

Measure-Like Configurations: An Example at the Borderline of Branched Configurations and Dynamics on Probability Spaces

Dynamics on probability space is a fast-growing subject and is shown to give rise to branched geodesics [11]. Following the same procedure as for branched topological foliations, we show how a restricted space fits with particular goals. The goals described here are linked with image recognition for the configuration space \( \Gamma^1 \) above, and to uncompressible fluid dynamics when we equip \( \varphi \) of \( \varphi \) with the dierology \( P_{\alpha} \) of constant volume above. Let \( C^0_{\alpha} \) be the set of compactly supported \( \mathbb{R} \)-valued smooth maps on \( N \). We define the relation of equivalence \( R \) by:
\[ f \in \varphi \iff \sup (f) = \sup (g). \]

Let us first give the definition of the set of 1-configurations:
\[ \Gamma^1_{\alpha} (N) = C^0_{\alpha} (N) / R \]

**Definition 4.1:** We set

Such a space is not a manifold, but we show that it carries a natural dierology. \( C^0_{\alpha} (N) \) is a topological vector space, and hence carries a natural dierology \( P_{\alpha} \). We define the following:

**Definition 4.2:** Let \( P_{\alpha} \subset P_{\alpha} \) be the set of \( P_{\alpha} \)-plots \( p \) \( p \rightarrow C^0_{\alpha} (N) \) such that, for any open subset \( A \) with compact closure \( A \) of \( N \), for any open subset \( O \) of \( O \) such that \( O \subset O \), if
\[ \text{p}(O)(A) = 0 \]
the map
\[ x \in O \rightarrow \text{Vol}(\sup (p(x)) \cap A) \]
is constant on \( O \), where \( \text{Vol} \) is the Riemannian volume.

This technical condition ensures that the volume of any connected component of the support of \( p(x) \) is constant. \( P_{\alpha} \) is obviously a dierology on \( C^0_{\alpha} (N) \), and we can state

**Proposition 4.3:** Let \( \varphi = \varphi \rightarrow \Gamma^1_{\alpha} (N) / R \) is a dierological space.

The proof is a straightforward application of Definition 4.12. It seems difficult to give this space a structure of Frölicher space, or even a natural topology except the topology of vague convergence of measures, which is not the topology induced by the dierology we have defined. As a consequence, we can only state that the well-defined configuration spaces \( \Gamma \) and \( \varphi \) are dierological spaces. The technical conditions of Definition 4.2 ensures that volume preserving is a consequence of smoothness with respect to \( P_{\alpha} \), and hence is particularly designed for (viscous) uncompressible fluid dynamics. A 1-configuration can have many connected components, and therefore branching effects are included in the definition of 1-configurations. One could understand \( n \)-configurations as the presence of \( n \) (non mixing) fluids.

**Appendix: Preliminaries on Differentiable Structures**

The objects of the category of -finite or infinite- dimensional smooth manifolds is made of topological spaces \( M \) equipped with a collection of charts called maximal atlas that enables one to make differentiable calculus. But there are some examples where a dierential calculus is needed whereas no atlas can be defined. To circumvent this problem, several authors have independently developed some ways to define dierentiation without defining charts. We use here three of them. The first one is due to Souriau [12], the second one is due to Sikorski, and the third one is a setting closer to the setting of dierentiable manifolds is due to Frölicher (see e.g. Cherenack P et al. [13] for an introduction on these two last notions).

In this section, we review some basics on these three notions.

**Souriau’s Dierological Spaces, Sikorski’s Dierential Spaces, Frölicher Spaces**

**Definition 4.4:** Let \( X \) be a set.

A plot of dimension \( p \) (or \( p \)-plot) on \( X \) is a map from an open subset \( O \subset \mathbb{R}^n \) to \( X \).

A dierology on \( X \) is a set \( P \) of plots on \( X \) such that, for all \( p \in P \), - any constant map \( \mathbb{R}^n \rightarrow X \) is in \( P \).

Let \( x \) be an arbitrary set; let \( f : \bigcup_{\alpha} O_{\alpha} \rightarrow X \) be a family of maps that extend to a map \( f : \bigcup_{\alpha} O_{\alpha} \rightarrow X \) if \( f : \bigcup_{\alpha} O_{\alpha} \subset P \), then \( f \in \text{EP} \) - (chain rule) Let \( f \) be a map on \( \bigcup_{\alpha} O_{\alpha} \) open subset of \( \mathbb{R}^n \) and \( g \) a smooth map (in the usual sense) from \( O' \) to \( O \). Then, \( f \cdot g \in \text{EP} \).

If \( P \) is a dierology \( X \), \( (X,P) \) is called dierological space. Let \( (X,P) \) be a dierological space, a map \( f : X \rightarrow X \) is dierentiable (=smooth) if and only if \( f \in \text{EP} \).

**Remark:** Notice that any dierological space \( (X,P) \) can be endowed with the weaker topology such that all the maps that are in \( P \) are continuous. But we prefer to mention this only for memory as well as other questions that are not closely related to our construction, and stay closer to the goals of this paper. Let us now define the Sikorski’s dierential spaces. Let \( X \) be a Hausdorff topological space.

**Definition 4.5:** A (Sikorski’s) dierential space is a pair \( (X,F) \) where \( F \) is a family of functions \( X \rightarrow \mathbb{R} \) such that
- the topology of \( X \) is the initial topology with respect to \( F \)
- for any \( n \in N \), for any smooth function \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \), for any \( (f_1,\ldots,f_n) \in P \), \( \phi \circ (f_1,\ldots,f_n) \in F \).

Let \( (X,F) \) and \( (X',F') \) be two dierential spaces, a map \( f : X \rightarrow X' \) is dierentiable (=smooth) if and only if \( F \circ f \in F' \).
We now introduce Frölicher spaces.

**Definition 4.6:** A Frölicher space is a triple \((X,F,C)\) such that \(-C\) is a set of paths \(\mathbb{R} \to X\)
- A function \(f : X \to \mathbb{R}\) is in \(F\) if and only if for any \(c \in C\), \(f \circ c \in C'(\mathbb{R},\mathbb{R})\);
- A path \(c : X \to X\) is in \(C\) (i.e. is a contour) if and only if for any \(f \in F\), \(f \circ c \in C'(\mathbb{R},\mathbb{R})\).

Let \((X,F,C)\) and \((X',F',C')\) be two Frölicher spaces, a map \(f : X \to X'\) is differentiable (=smooth) if and only if \(f' \circ c \in C'(\mathbb{R},\mathbb{R})\) for any \(c \in C\).

Any family of maps \(F_x\) from \(X\) to \(\mathbb{R}\) generate a Frölicher structure \((X,F,C)\), setting \([14]\):

- \(-C = \{c : \mathbb{R} \to X\} \text{ such that } F_x \circ c \in C'(\mathbb{R},\mathbb{R})\}
- \(-F = \{f : X \to \mathbb{R}\} \text{ such that } f \circ c \in C'(\mathbb{R},\mathbb{R})\}

One easily see that \(F \subset C\). This notion will be useful in the sequel to describe in a simple way a Frölicher structure.

A Frölicher space, as a differential space, carries a natural topology, which is the pull-back topology of \(\mathbb{R}\) via \(f\). In the case of a finite dimensional differentiable manifold, the underlying topology of the Frölicher structure is the same as the manifold topology. In the infinite dimensional case, these two topologies differ very often.

In the three previous settings, we call \(X\) a differentiable space, omitting the structure considered. Notice that, in the three previous settings, the sets of differentiable maps between two differentiable spaces satisfy the chain rule. Let us now compare these three settings: One can see (see e.g. \([13]\)) that we have the following, given at each step by forgetful functors:

- smooth manifold \(\Rightarrow\) Frölicher space \(\Rightarrow\) Sikorski differential space

Moreover, one remarks easily from the definitions that, if \(f\) is a map from a Frölicher space \(X\) to a Frölicher space \(X'\), \(f\) is smooth in the sense of Frölicher if and only if it is smooth in the sense of Sikorski.

One can remark, if \(X\) is a Frölicher space, we define a natural diffeology on \(X\) by Magnot \([15]\):

\[
P(F) = \prod_{p \in P} \{f : p\text{-paramatization on } X; F \circ f \in C'(\mathbb{R},\mathbb{R}) \text{ (in the usual sense)}\}.
\]

With this construction, we get also a natural difieology when \(X\) is a Frölicher space. In this case, one can easily show the following:

**Proposition 4.7:** Let \((X,F,C)\) and \((X',F',C')\) be two Frölicher spaces. A map \(f : X \to X'\) is smooth in the sense of Frölicher if and only if it is smooth for the underlying difeologies \([15]\).

Thus, we can also state:

- smooth manifold \(\Rightarrow\) Frölicher space \(\Rightarrow\) Diffeological space

**Cartesian Products**

The category of Sikorski differential spaces is not cartesianly closed, see e.g. \([13]\). This is why we prefer to avoid the questions related to cartesian products on differential spaces in this text, and focus on Frölicher and difeological spaces, since the cartesian product is a tool essential for the definition of configuration spaces.

In the case of difeological spaces, we have the following \([12,16-19]\):

**Proposition 4.8:** Let \((X,P)\) and \((X',P')\) be two difeological spaces. We call product difeology on \(XX'\) the difeology \(P \times P'\) made of plots \(g: O \to XX'\) that decompose as \(g = f \times f'\), where \(f: O \to X \in P\) and \(f': O \to X' \in P'\).

Then, in the case of a Frölicher space, we derive very easily, compare with e.g. Kriegl A et al. \([14]\):

**Proposition 4.9:** Let \((X,F,C)\) and \((X',F',C')\) be two Frölicher spaces, with natural difeologies \(P\) and \(P'\). There is a natural structure of Frölicher space on \(XX'\)which contours \(C \times C\) are the 1-plots of \(P \times P'\).

We can even state the following results in the case of infinite products.

**Proposition 4.10:** Let \(I\) be an infinite set of indexes, that can be uncoutable.

(adapted from \([21]\)) Let \(\{(X_i,P_i)\}_i\) be a family of difeological spaces indexed by \(I\). We call product difeology on \(\prod_{i} X_i\) the difeology \(\prod_{i} P_i\) made of plots \(g: O \to \prod_{i} X_i\) that decompose as \(g = \prod_{i} g_i\), where \(g_i \in P_i\). This is the biggest difeology for which the natural projections are smooth.

Let \(\{(X,F,C_i)\}_i\) be a family of Frölicher spaces indexed by \(I\), with natural difeologies \(P_i\). There is a natural structure of Frölicher space \(\prod_{i} X_i\) which contours \(\prod_{i} X_i\) are the 1-plots of \(\prod_{i} P_i\). A generating set of functions for this Frölicher space is the set of maps of the type:

\[
\phi \circ \prod_{i} f_i,
\]

where \(J\) is a finite subset of \(I\) and \(\phi\) is a linear map \(\mathbb{R}^{|J|} \to \mathbb{R}\).

**Proof:** By definition, following \([12,20]\), \(\prod_{i} P_i\) is the biggest difeology for which natural projections are smooth. Let \(g: O \to X\) be a plot.

\[
g \in \phi \circ \prod_{i} p_i \Leftrightarrow p_i \circ g \in P_i
\]

where \(p_i\) is the natural projection onto \(X_i\), which gets the result.

With the previous point and Proposition 4.7, we get the family of contours of the product Frölicher space.

**Push-Forward, Quotient and Trace**

We give here only the results that will be used in the sequel.

**Proposition 4.11:** Let \((X,P)\) be a difeological space, and let \(X'\) be a set. Let \(f : X \to X'\) be a surjective map. Then, the set \([12,21]\).

\[
f(P) = \{u \text{ such that } u \text{ restricts to some maps of the type } f \circ p, p \in P\}
\]

is a difeology on \(X'\), called the push-forward dfeology on \(X'\) by \(f\).

We have now the tools needed to describe the dfeology on a quotient:

**Proposition 4.12:** Let \((X,P)\) be a difeological space and \(R\) an equivalence relation on \(X\). Then, there is a natural dfeology on \(X/R\), noted by \(P/R\), defined as the push-forward dfeology on \(X/R\) by the quotient projection \(X \to X/R\).

Given a subset \(X_0 \subset X\), where \(X\) is a Frölicher space or a difeological space, we can define on trace structure on \(X_0\) induced by \(X\).
If $X$ is equipped with a diffeology $P$, we can define a diffeology $P_0$ on $X_0$ setting

$$P_0 = \{ p \in P \text{ such that the image of } p \text{ is a subset of } X_0 \}.$$ 

If $(X, F, C)$ is a Frolicher space, we take as a generating set of maps $F$ on $X_0$ the restrictions of the maps $f \in F$. In that case, the contours (resp. the induced diffeology) on $X_0$ are the contours (resp. the plots) on $X$ which image is a subset of $X_0$.

References