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# **Short Communication**

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# Geometric Proof of the Sum of Geometric Series

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### Introduction

The well-known formula for the sum of the geometric series is

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

For arbitrary -1 < q < 1. Among analytic proofs, geometric proofs were also given for this formula, see [1], mostly for 0 < q < 1. Now we prove that for any 0 < q < 1

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

holds in the 'Positive case' and

$$s' = \sum_{k=1}^{\infty} (-1)^{k-1} q^k = \frac{q}{1-q}$$

in the 'Alternating case'.

#### **Positive case**

As in Figure 1, we do the following process.

**Step 1**: Take a unit square  $S_i$  and take rectangles  $Q_i$  with area  $A_{Q_1} = q$  and  $Q_2$  with  $A_{Q_2} = q_2$ . We also

take 'adjunct' rectangles  $R_1$  with  $A_{R_1} = \frac{1-2q}{q} A_{Q1}$  and  $R_2$  with  $A_{R_2} = \frac{1-2q}{q} A_{Q2}$ .

We get a remaining square  $S_2$  of side length q.

**Step 2:** We repeat the actions of the previous step for  $S_2$ , but we reduce the rectangles by a scale factor of q. Then we get  $Q_3$ ;  $Q_4$ ;  $R_3$ ;  $R_4$  with  $A_{Q_4} = q^4$ ,  $A_{Q_4} = q^4$ ,  $A_{R_3} = \frac{1-2q}{q}A_{Q_3}$ ,  $A_{R_5} = \frac{1-2q}{q}A_{Q_5}$  and a remaining square  $S_3$  of side length  $q^2$ .

**General step** *k*: We repeat the actions of the previous step for  $S_{k^2}$  but we reduce the rectangles by a scale factor of *q*.

We get that the area of unit square  $S_1$  is

$$1 = \sum_{k=1}^{\infty} A_{Q_k} + \sum_{k=1}^{\infty} A_{R_k} = s + \frac{1-2q}{q} S = \frac{1-q}{q} s,$$
  
Hence  $S = \frac{q}{1-q}$   
Alternating case

As in Figure 2, we do the following process.

Received: March 28, 2018 Accepted: July 10, 2018 Published:Septembr 8, 2018



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**Step 1:** Take a unit square *S* and take rectangles  $Q_1^+$  with  $A_{Q_1^+} = q$ and  $Q_2^-$  with  $A_{Q_2^-} = q^2$ . We also take 'adjunct' square  $R_1^+$  of side length 1 with  $A_{R_1^+} = \frac{1}{q} A_{Q_1^+}$  and rectangle  $R_2^-$  of side lengths q and 1 with  $A_{R_2^-} = \frac{1}{q} A_{Q_1^+}$ . (Plus and minus signs indicate the signs of the areas of rectangles within the final sum.)

**Step 2:** We repeat the actions of the previous step for square  $Q_2^-$  of side length q, but we reduce the rectangles by a scale factor of q. Then we get  $Q^+, P^+, P^-$  with  $A_{QP} = q^4, A_{QP} = q^4, A_{PP} = \frac{1}{2}A_{PP}$ .

$$\begin{array}{l} \text{Inen we get } Q_{1}^{+}, R_{3}^{+}, R_{3}^{+}, R_{4}^{-} \text{ with } A_{\underline{Q}_{4}^{-}} - q \ , \ A_{\underline{Q}_{4}^{-}} - q \ , \ A_{\underline{R}_{5}^{+}} = -A_{\underline{Q}_{4}} \\ A_{\underline{R}_{4}^{-}} = \frac{1}{q} A_{\underline{Q}_{4}} \end{array}$$

**General step** *k*: We repeat the actions of the previous step for  $Q_{2k-2}^-$ , but we reduce the rectangles by a scale factor of *q*.

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We get that the area of unit square  $S_1^{'}$  is

$$\begin{split} &1 = \sum_{k=1}^{\infty} \left( A_{Q_{2k-1}^{+}} - A_{Q_{2k}^{-}} + A_{R_{2k-1}^{+}} - A_{R_{2k}^{-}} \right) = \sum_{k=1}^{\infty} (-1)^{k-1} q^{k} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{q} q^{k} \\ &= s + \frac{1}{q} s = \frac{1+q}{q} s \end{split}$$

Hence 
$$s' = \frac{q}{1+q}$$
.

#### References

1. Nelsen RB (1993) Proofs without words: Exercises in visual thinking. MAA.

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