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Upper Semicontinuous Representability of Maximal Elements for Non total Preorders on Compact Spaces

Gianni Bosi[1]*, Paolo Bevilacqua[2] and Magali Zuanon[3]

Abstract

We discuss the possibility of determining all the maximal elements of a preorder on a compact topological space by maximizing all the functions in a suitable family of upper semicontinuous order-preserving functions.

Keywords

Preorder; Order-preserving function; Weak utility; Maximal element; Upper semicontinuous function

Introduction

White's theorem [1] is important since, for every maximal element x_o relative to a preorder $\prec$ on set X, it guarantees the existence of an order-preserving function u on the preordered set $(X, \precsim)$ attaining its maximum at x_o, provided that an order-preserving function u' on $(X, \precsim)$ exists. So, at least theoretically, every maximal element is obtained by maximizing a real-valued order-preserving function. When this happens, it is clear that every maximal element is potentially optimal in the sense of Podinovski et al. [2] (i.e., for every maximal element there exists a total preorder extending the original preorder with respect to which such maximal element is best preferred).

In this paper, we generalize White's theorem to the upper semicontinuous case. This means that we present conditions on a preorder $\precsim$ on a topological space (X,τ) under which, for every maximal element x_o relative to $\prec$, there exists an upper semicontinuous order preserving function u on the preordered topological space $(X,\tau, \precsim)$ attaining its maximum at x_o, provided that an upper semicontinuous order-preserving function u' on $(X,\tau, \precsim)$ exists. It should be noted that Bevilacqua et al. [3] already characterized the property according to which every maximal element relative to a preorder on a compact topological space can be obtained by maximizing a transfer weakly upper continuous weak utility for its strict part (see the generalization of Weierstrass Theorem presented by Tian et al. [4]).

It is clear that these results are important due to the well-known fact that every upper semicontinuous (more generally transfer weakly upper continuous) function attains its maximum on a compact topological space, and the nearly obvious consideration that a point x_o at which an order-preserving function u for a preorder (or, more generally, a weak utility for its strict part) attains its maximum is also a maximal element for $\precsim$.

*Corresponding author: Gianni Bosi, Bruno de Finetti Department of Economics, Business, Mathematics and Statistics (DEAMS), University of Trieste, Italy, Tel: +39-040-558-7115; E-mail: gianni.bosi@deams.units.it

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Notation and preliminaries

Let X be a nonempty set (decision space). A binary relation $\precsim$ on X is interpreted as a weak preference relation, and therefore, for any two elements $x, y \in X$, the scripture "$x \precsim y$" has to be thought of as "*the alternative $y \in X$ is at least as preferable as $x \in X$*". As usual, $\prec$ denotes the strict part of a binary relation $\precsim$ (i.e., for all $x, y \in X$, $x \prec y$ if and only if $(x \precsim y)$ and not $(y \precsim x)$. A preorder is a reflexive and transitive binary relation. An anti-symmetric preorder $\precsim$ is referred to as an order. Furthermore, $\sim$ stands for the indifference relation (i.e., for all $x, y \in X$, $x \sim y$ if and only if $(x \precsim y)$ and $(y \precsim x)$.We have that $\sim$ is an equivalence relation on X whenever $\precsim$ is a preorder.

For every $x \in X$, we set

$l(x)=\{z \in X : z \prec x\}$ $i(x)=\{ z \in X : x \prec z\}$

Given a preordered set $(X, \precsim)$, a point $x_o \in X$ is said to be a maximal element for $\precsim$ if for no $z \in X$ it occurs that $x_o \prec z$. In the sequel we shall denote by $X_M^{\precsim}$ the set of all the maximal elements of a preordered set $(X, \precsim)$. Please observe that $X_M^{\precsim}$ can be empty.

We recall that a function $u: (X, \precsim) \to (R, \leq)$ is said to be

i. isotonic or increasing if, for all $x, y \in X$, $x \precsim y \to u(x) \leq u(y)$;

ii. a weak utility for $\prec$ if, for all $x, y \in X$, $x \prec y \to u(x) < u(y)$;

iii. Strictly isotonic or order-preserving if it is both isotonic and a weak utility for $\prec$.

Strictly isotonic functions on $(X, \precsim)$ are also called Richter-Peleg representations of $\precsim$ in the economic literature (see e.g. Richter et al. [5] and Peleg et al. [6])

A preorder $\precsim$ on a topological space (X, τ) is said to be

i. upper semicontinuous if, for all $x, y \in X$, $i(x)=\{ z \in X : x \precsim z\}$ is a closed subset of X for every $x \in X$;

ii. Quasi upper semicontinuous if there exists an upper semicontinuous preorder $\precsim$ on (X , τ) such that $\prec \subset \prec$.

An upper semicontinuous preorder Ward et al. [7] or more generally a quasi-upper semicontinuous preorder Bosi et al. [8, Theorem 3.1] $\precsim$ on a compact topological space (X , τ) admits a maximal element. As usual, for a real-valued function u on X, we denote by *arg max u* the set of all the points $x \in X$ such that u attains its maximum at x (i.e., $\arg\max u=\{z \in X:(u(z)\leq u(x)) \forall z \in X\}$

If τ is a topology on a set X, and $\precsim$ is a preorder on X, then the triplet $(X, \tau, \precsim)$ will be referred to as a topological preordered space. The (natural) (interval) topology on the real line R will be denoted by τ_{nat} .

Finally, we recall that a real-valued function u on a topological space (X , τ) is said to be upper semicontinuous if $u^{-1}(]-\infty, \alpha[)=\{x \in X:$

$u(x) < \alpha\}$ is an open set for all $\alpha \epsilon R$. A very well know result guarantees that every upper semicontinuous real-valued function u on a compact topological space (X, τ) attains its maximum.

Maximal elements of preorders from maximization of upper semicontinuous functions

The following theorem was proved by White et al. [1]. Given any maximal element x_o relative to a preorder $\precsim$ on a set X, it guarantees the existence of some order-preserving function u attaining its maximum at x_o, provided that an order-preserving function $u': (X, \precsim) \to (R, \leq)$ exists. Therefore, in order to determine all the maximal elements of a preorder $\precsim$ on a set X, the agent maximizes all the functions u in a family **U** of bounded order-preserving functions for $\precsim$. Needless to say, this is a very important opportunity, at least theoretically.

Theorem: (White et al. [1]): Let $(X, \precsim)$ be a preordered set and assume that there exists an order-preserving function $u':(X, \precsim) \to (R, \leq)$. If $X_M^{\precsim}$ is nonempty, then for every $x_o \in X_M^{\precsim}$ there exists an order-preserving function $u:(X, \precsim) \to (R, \leq)$. Such that *arg max* $u=[x_0]=\{z \in X: z \sim x_o\}$.

We now present a generalization of the above theorem to the "*upper semicontinuous case*".

Theorem: Let $(X, \tau, \precsim)$ be a topological preordered space, and assume that $X_M^{\precsim}$ is nonempty. Consider an element $x_o \epsilon X_M^{\precsim}$. Then the following conditions are equivalent:

i. There exists an upper semicontinuous order-preserving function $u':(X, \tau, \precsim) \to (R, \tau_{nat}, \leq)$ and $[x_0]=\{z \in X: z \sim x_o\}$ is a closed subset of X;

ii. There exists an upper semicontinuous order-preserving function $u:(X, \tau, \precsim) \to (R, \tau_{nat}, \leq)$ such that *arg max* $u=[x_0]=\{z \in X: z \sim x_o\}$.

Proof: Consider a topological preordered space $(X, \tau, \precsim)$.

(i) $\Rightarrow$ (ii) Let u' be an upper semicontinuous order-preserving function on (X, τ, X). Without loss of generality, we can assume u' to be bounded. Consider a point $x_o \in X_M^{\precsim}$ and define the real-valued function u *on* X as follows for any choice of a positive real δ:

$u'(x)$ *if not* $(x \sim xo)$

$u(x)=\{$

$\sup u'(x)+\delta$ *if* $(x \sim xo)$

White et al. [1, Theorem 1] proved that the above function u is order-preserving for $\precsim$ as soon as u' is order-preserving for $\precsim$. For the sake of completeness, let us recall here the arguments supporting this consideration. It is clear that $u'(x) \leq u(x)$ for every $x \in X$. In order to show that u is increasing with respect to $\precsim$, consider any two points $x, y \in X$ such that $x \precsim y$. If $y \sim x_o$, then it is clear that $u(x) \leq u(x)$ from the definition of u, On the other hand, if not$(y \sim x_o)$, then it must be also not$(x \precsim x_o)$, since $x_o \sim x \precsim y$ would imply$(x_o \precsim y)$,, and in turn $x_o \sim y$ due to the fact that x_o is a maximal element relative to $\precsim$. Hence, since neither $x \sim x_o$ nor $y \sim x_o$, we have that $u(x)=u'(x)\leq u'(y)= u(y)$ from the definition of u and the fact that u' is increasing with respect to $\precsim$. In order to show that u is a weak utility for $\prec$, consider any two points $x, y \in X$ such that $x \prec y$. Then we have that not$(x \sim x_o)$, since $x_o \sim x \prec y$ implies that $x_o \sim y$ (a contradiction, since x_o is assumed to be a maximal element for $\precsim$). Therefore, from the definition of u and the fact that u' is a weak utility for $\prec$, we have that $u(x)=u'(x)<u'(y)\leq u(y)$, which obviously implies that $u(x)<u(y)$. Further, u attains its maximum at x_o and actually, since δ is a positive real, it is clear that (i) *arg max* $u=[x_0]=\{z \in X: z \sim x_o\}$ Therefore, it only remains to show that under our assumptions u is upper semicontinuous. It is clear that u is upper semicontinuous at every point $x \in X$ such that $x \sim x_o$. Therefore, consider any point $x \in X$ such that not $x \sim x_o$ and $\alpha \in R$ such that $u(x) < \alpha$. We can limit our considerations to the case when $\alpha \leq \sup u'(x) + \delta$. Since in this case $u(x)=u'(x)$, from upper semi continuity of u' there exists an open set U_x' containing x such that $u'(z) < \alpha$ for every $z \epsilon U_x'$. Since $[x_o]$ is closed, we have that $U_x = U_x' - [x_o]$ is an open set containing x such that $u'(z)=u(z) < \alpha$ for every $z \epsilon U_x'$. Hence, u is an upper semicontinuous function.

(ii) $\Rightarrow$ (i) Assume that there exists an upper semicontinuous order-preserving function $u:(X, \tau, \precsim) \to (R, \tau_{nat}, \leq)$ such that *arg max* $u=[x_0]=\{z \in X: z \sim x_o\}$. If $[x_0]$ is not closed, then there exists an element $z \epsilon X-[x_0]$ such that $U_z \cap [x_0] \neq \Phi$ for every neighborhood U_z of the element z. But this contradicts the fact that u is upper semicontinuous, since in this case $u^{-1}(]-\infty, u(x_o)[)$, an open set containing z, should contain an element $z' \in [x_0]$, for which $u(z')=u(x_o)$. This consideration completes the proof.

Remark: It is clear that Theorem 3.2 generalizes White's theorem, due to the fact that these two results precisely coincide when we consider the discrete topology τ on X.

Since in order to determine a maximal element relative to preorder $\precsim$ on a set X it suffices to maximize a weak utility for the strict part $\prec$ of $\precsim$, the following corollary can be considered as useful. Indeed, the reader can easily verify that the implication "*(i)* $\Rightarrow$ *(ii)*" in Theorem 3.2 is still valid if one considers weak utilities for $\prec$ instead of order-preserving functions u' for X

Corollary: Let $(X, \tau, \precsim)$ be a topological preordered space with τ a compact topology. If there exists an upper semicontinuous weak utility u' for $\prec$, and $[x]=\{z \in X: z \sim x\}$ is a closed set for all $x \epsilon X_M^{\precsim}$, then for every $x_o \in X_M^{\precsim}$ there exists an upper semicontinuous weak utility u for $\prec$ such that *arg max* $[x]=\{z \in X: z \sim x_o\}$.

Bosi et al. [8, Theorem 2.11] proved that there exists an upper semicontinuous weak utility for the strict part $\prec$ of a quasi-upper semicontinuous preorder $\precsim$ on a second countable (i.e., with a countable base) topological space(X, τ). Hence, we get the following corollary.

Corollary: Let $(X, \tau, \precsim)$ be a topological preordered space. Assume that τ is a compact and second countable topology, and that $\precsim$ is quasi upper semicontinuous. If $[x]=\{z \in X: z \sim x\}$ is a closed set for all $x \epsilon X_M^{\precsim}$, then for every $x_o \in X_M^{\precsim}$ there exists an upper semicontinuous weak utility u for $\prec$ such that *arg max* $[x]=\{z \in X: z \sim x_o\}$.

Conclusion

In this paper, following the spirit of a theorem of White et al. [1], we have presented some results concerning the representation of the set of all maximal elements of a preorder on a compact topological space by means of the maximization of all functions in a suitable family of bounded upper semicontinuous order-preserving functions. The more delicate problem of characterizing the possibility of representing all the maximal elements for a preorder on a compact topological space by means of the maximization of finitely many upper semicontinuous order-preserving functions will be hopefully considered in a future paper.

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Author Affiliation Top

[1]Bruno de Finetti Department of Economics, Business, Mathematics and Statistics (DEAMS), University of Trieste, Italy

[2]Department of Engineering and Architecture, University of Trieste, Italy

[3]Department of Economy and management, University of Brescia, Italy