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Enlarging the Radius of Convergence for the Halley Method to Solve Equations with Solutions of Multiplicity under Weak Conditions

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Abstract

The objective of this paper is to enlarge the ball of convergence and improve the error bounds of the Halley method for solving equations with solutions of multiplicity under weak conditions.

Keywords

Halley's method; Solutions of multiplicity; Ball convergence; Derivative; Divided difference

Introduction

Many problems in applied sciences and also in engineering can be written in the form like

$$f(x) = 0, \tag{1.1}$$

Using mathematical modeling, where $f : D \subseteq \square \rightarrow \square$ is sufficiently many times differentiable and D is a convex subset in $\square$. In the present study, we pay attention to the case of a solution p of multiplicity $m>1$; namely, $f(x_*)=0$, $f^{(i)}(x_*)=0$ for $i=1,2,\ldots m-1$, and $f^{(m)}(x_*)\neq 0$

The determination of solutions of multiplicity m is of great interest. In the study of electron trajectories, when the electron reaches a plate of zero speed, the function distance from the electron to the plate has a solution of multiplicity two. Multiplicity of solution appears in connection to Van Der Waals equation of state and other phenomena. The convergence order of iterative methods decreases if the equation has solutions of multiplicity m. Modifications in the iterative function are made to improve the order of convergence. The modified Newton's method (MN) defined for each $n=0,1,2,..$

$$X_n+1=x_n-mf(x_n)-1f(x_n), \tag{1.2}$$

Where $x_0\in D$ is an initial point is an alternative to Newton's method in the case of solutions with multiplicity m that converges with second order of convergence.

A method with third order of convergence is defined by modified Halley method (MH) [4]

$$x_{n+1} = x_n - \left(\frac{m+1}{2m} f'(x_n) - \frac{f''(x_n)\,\mathrm{f}(x_n)}{2f'(x_n)}\right)^{-1} f(x_n) \tag{1.3}$$

Method (1.3) is an extension of the classical Halley's method of the third order. Other iterative methods of high convergence order can be found in [1-15] and the references therein.

Let $U(p,\lambda) := \{xU_1: |x-p|<\lambda$ denote an open ball and $\bar{U}(p,\lambda)$ denote its closure. It is said that $U(p,\lambda) \subseteq D$ is a convergence ball for an iterative method, if the sequence generated by this iterative method converges to p; provided that the initial point $x_0 \in U(p,\lambda)$. But how close x_0 should be to x_* so that convergence can take place. Extending the ball of convergence is very important, since it shows the difficulty; we confront to pick initial points. It is desirable to be able to compute the largest convergence ball. This is usually depending on the iterative method and the conditions imposed on the function f and its derivatives. We can unify these conditions by expressing them as:

$$||(f^{(m)}(x_*))^{-1}(f^{(m+1)}(x)||\leq\varphi 0(||x- x_*||) \tag{1.4}$$

$$||(f^{(m)}(x_*))^{-1}(f^{(m+1)}(x)- f^{(m+1)}(y)||\leq\varphi\,(||x- y||) \tag{1.5}$$

for all $x, y \in D$; where φ_0; φ: $\square_+$ U $\{0\} \rightarrow \square_+$ U $\{0\}$ are continuous and nondecreasing functions satisfying $\varphi(0)=0$: If, m 1; $\varphi_0\,(t)=0$ and

$$\varphi(t)=\mu t^q,\ \mu_0>0,\ \mu>0,\ q\in [0,1],$$

Then, we obtain the conditions under which the preceding methods were studied [1-17]. However, there are ceases where even (1.6) does not hold (see Example 4.1). Moreover, the smaller functions φ_0, φ are chosen, the larger the radius of convergence becomes. The technique, we present next can be used for all preceding methods as well as in methods where $m=1$: However, in the present study, we only use it for MH. This way, in particular, we extend the results in [4,5,12,13,16,17].

The rest of the paper is structured as follows. Section 2 contains some auxiliary results on divided differences and derivatives. The ball convergence of MH is given in Section 3. The numerical examples in the concluding Section 4.

Auxiliary Results

In order to make the paper as self-contained as possible, we restate some standard definitions and properties for divided differences [4,13,16,17].

Definition: The divided differences $f[y_0, y_1,\ldots, y_k]$, on $k+1$ distinct points $y_0, y_1,\ldots,y_k$ of a function $f(x)$ are defined by

$$f[y_0]= f(y_0)$$

$$f[y_0,y_1] = \frac{f[y_0]-f[y_1]}{y_0-y_1}$$

$$f[\mathrm{y}_0,y_1,\ldots,yk] = \frac{f[y_0,\mathrm{y}_1,\ldots,\mathrm{yk}-1]-f[y_0,y_1,\ldots,\mathrm{yk}]}{y_0-yk} \tag{2.1}$$

If the function f is sufficiently differentiable, then its divided differences $f[y_0, y_1,\ldots, y_k]$ can be defined if some of the arguments y_i

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coincide. for instance, if $f(x)$ has k-*th* derivative at y_0; then it makes sense to define

$$f[\underbrace{y_0, y_1, \ldots, yk}_{k+1}] = \frac{f^{(k)}(y_0)}{k!}$$

Lemma: The divided differences $f[y_0, y_1, \ldots, y_k]$ are symmetric functions of their arguments,i.e., they are invariant to permutations of the $y_0, y_1, \ldots, y_k$.

Lemma: If the function f has k-*th* derivative, and $f^{(k)}(x)$ is continuous on the interval $I_x=[min(y_0, y_1, \ldots, y_k), max(y_0, y_1, \ldots, y_k)]$, then

$$f[y_0, y_1, \ldots, y_k] = \int_0^1 \cdots \int_0^1 \theta_2^{k-1}\theta_1^{k-1} \ldots \theta^{k-1} f^{(k)}(\theta)\,\mathrm{d}\theta_1 \ldots \theta_k \tag{2.3}$$

Where $\theta = y_0 + (y_1 - y_0)\theta_1 + (y_2 - y_1)\,\theta_1\theta_2 + \ldots + (y_k - y_{k-1})\,\theta_1 \ldots \theta_k$

Lemma: If the function f has $(k+1)$-*th* derivative, then for every argument x; the following formulae holds

$$f(x) = f[v_0] + f[v_0, v_1](x - v_0) + \ldots + f[v_0, v_1, \ldots, v_k]\,(x - v_0) \ldots (x - v_k) + f[v_0, v_1, \ldots, v_k, x]\,\lambda(x), \tag{2.4}$$

Where

$$\lambda(x) = (x - v_0)\,(x - v_1) \ldots (x - v_k) \ldots \tag{2.5}$$

Lemma: Assume the function f has continuous $(m + 1)$-*th* derivative, and x_* is a zero of multiplicity m; we define functions g_0, g and g_1 as

$$g_0(x) = f[\underbrace{x_*, x_*, \ldots, x_*}_{m}, x, x],\ g(x) = f[\underbrace{x_*, x_*, \ldots, x_*}_{m}, x] \qquad g_1(x) = f[\underbrace{x_*, x_*, \ldots, x_*}_{m}, x, x] \tag{2.6}$$

Then,

$$g'(x) = g_0(x),\ g''(x)2g_1(x). \tag{2.7}$$

Lemma: If the function f has an $(m+1)$-*th* derivative, and x_* is a zero of multiplicity m, then for every argument x, the following formulae hold

$$f(x) = f[\underbrace{x_*, x_*, \ldots, x_*}_{m}, x, x](x - x_*)^{m} = g(x)(x - x_*)^{m} \tag{2.8}$$

$$f'(x) = f[\underbrace{x_*, x_*, \ldots, x_*}_{m}, x, x](x - x_*)^{m}$$
$$+ mf[\underbrace{x_*, x_*, \ldots, x_*}_{m}, x, x](x - x_*)^{m-1} \tag{2.9}$$
$$= g_0(x)(x - x_*)^{m} + mg(x)(x - x_*)^{m-1}$$

And

$$f''(x) = 2f[\underbrace{x_*, x_*, \ldots, x_*}_{m}, x, x, x](x - x_*)^{m}$$
$$+ 2mf[\underbrace{x_*, x_*, \ldots, x_*}_{m}, x, x](x - x_*)^{m-1}$$
$$+ m(m-1)f[\underbrace{x_*, x_*, \ldots, x_*}_{m}, x, x](x - x_*)^{m-2}$$
$$= 2g_1(x)(x - x_*)^{m} + 2mg_1(x)(x - x_*)^{m-1}$$
$$+ m(m-1)g(x)(x - x_*)^{m-2}$$

where $g_0(x)$; $g(x)$ and $g_1(x)$ are defined previously.

Local Convergence

It is convenient for the local convergence analysis that follows to define some real functions and parameters. Define the function ψ_0 on $_+ U\{0\}$ by

$$\psi_0 = \frac{\varphi_0(t)t}{m+1} - 1$$

We have $_0(0) = -1 < 0$ and $\psi_0(t) > 0$. Suppose

$\psi_0(t)t \rightarrow$ a positive number of $+\infty$

for sufficiently large t. It then follows from the intermediate value theorem that function ψ_0 has zeros in the interval $(0, +\infty)$: Denote by ρ_0 the smallest such zero. Define functions $\varphi_0^{(m)}, \varphi^{(m)}, p, q, \psi$ on the interval $[0, \rho_0)$ by

$$\varphi_0^{(m)}(t) = m!\int_0^1 \cdots \int_0^1 \theta_1^{m}\theta_2^{m-1} \ldots \theta_m \varphi_0(\theta_1, \ldots, \theta_m t)\,d\theta_1 \ldots d\theta_{m+1}$$

$$\varphi^{(m)}(t) = m!\int_0^1 \theta_1^{m}\theta_2^{m-1} \ldots \theta_m \varphi(\theta_1, \ldots, \theta_{m-1}(1-\theta_m)\,d\theta_1 \ldots d\theta_{m+1}$$

$$g_0(t) = 2\mathrm{m}^2 - \frac{2\mathrm{m}^2}{m+1}\varphi(t)t - 2m\varphi_0^{(m)}(t)t - 2m\varphi^{(m)}(t)$$

$$g_1(t) = g_0(t) - 2\mathrm{m}^2 + \frac{2\mathrm{m}^2\varphi_0(t)t}{m+1} + 2m\varphi_0^{(m)}(t)t$$

And

$$\psi(t) = \frac{g_1(t)}{g_0(t)} - 1$$

We get that $g_0(0) = 2\mathrm{m}^2(1 - \frac{1}{m+1}) = \frac{2m^3}{m+1} > 0$ and $g_0(t) \rightarrow -\infty$ as $t \rightarrow \rho_0^-$.

Denote by r_0 the smallest zero of function g_0 in the interval $(0, \rho_0)$: Moreover, we get that $\psi_0(0) = -2 < 0$ and $\psi(t) \rightarrow +\infty$ as $t \rightarrow r_0$

Denote by r the smallest zero of function ψ on the interval $(0, r_0)$: Then, we have that for each $t \in [0, r)$

$0 \leq (t) < 1$

The local convergence analysis is based on conditions (A):

(A_1) Function $f: D \subseteq □ \rightarrow □$ is $(m+1)$ times differentiable and x_* is a zero of multiplicity m.

(A_2) Conditions (1.4) and (1.5) hold

(A_3) $\bar{U}(x_*, r) \subseteq D$, where the radius of convergence r is defined previously.

(A_4) Condition (3.1) holds

Theorem Suppose that the (A) conditions hold. Then, sequence $\{x_n\}$ generated for $x_0 \in U(x_*, r) - \{x_*\}$ by *MH* is well defined in $U(x_*, r)$, remains in $U(x_*, r)$ for each $n = 0, 1, 2\ldots$and converges to x_*

Proof. We base the proof on mathematical induction. Set $\delta_n = x_n - x_*$ and choose initial point $x_0 \in U(x_*, r) - \{x_*\}$. Using (1.2), (2.8), (2.9) and (2.10), we have in turn that

x_1

$$x_0 - \frac{f(x_0)}{\frac{m+1}{2m}f'(x_0) - \frac{f(x_0)f''(x_0)}{2f'(x_0)}}$$

$$x_0 - \frac{2mg(x_0)(g_0(x_0)\delta_0 + mg(x_0)\delta_0}{[(m+1)g_0^2(x_0) - 2\mathrm{m}g(x_0)g_1(x_0)]\delta_1^2 + 2mg_1(x_0)g(x_0)\delta_0 + 2m^2g^2(x_0)} \tag{3.3}$$

so

$$= \delta_0 - \frac{2mg(x_0)(g_0(x_0)\delta_0 + mg(x_0))\delta_0}{[(m+1)g_0^2(x_0) - 2\mathrm{m}g(x_0)g_0(x_0)]\delta_0^2 + 2mg_0(x_0)g(x_0)\delta_0 + 2m^2g^2(x_0)}$$

$$=\frac{[(m+1)g_1^2(x_0)-2\mathrm{mg}(x_0)g_1(x_0)]\delta_0^3}{[(m+1)g_0^2(x_0)-2\mathrm{mg}(x_0)g_1(x_0)]\delta_0^2+2mg_1(x_0)g(x_0)\delta_0+2m^2g^2(x_0)} \quad (3.4)$$

or

$$\delta_1=\frac{\alpha_0\delta_0^2}{\alpha_0\delta_0+2m\beta_0} \quad (3.5)$$

Where

$$a_n=(m+1)\left(g\left(x_n\right)\right)-1g_0^2(x_n)\delta_n-2mg_1(x_n)\delta_n(n=0,1,2,\ldots)$$

$$\beta_n=g_0\left(x_n\right)\delta_n+mg(x_n)(n=0,1,2,\ldots) \quad (3.6)$$

By (2.2) and (2.6), we can get

$$g_0(x_*)=f[\underbrace{x_*,x_*,\ldots x_*}_{m+1}]=\frac{f^{(m)}(x_*)}{m!} \quad (3.7)$$

and by the condition (1.4) and

$x_0\in U(x_*,r)\subseteq U(x_*,r_0)$ we obtain,

$$|1-(g(x_*))^{-1}g(x_0)|=|(g(x_*))^{-1}(g_0(x_*)-g(x_0))|$$

$$=|(g(x_*))^{-1}\frac{f^{(m+1)}(y_0)}{(m+1)!}\delta_0|$$

$$\leq\frac{\varphi_0(|\delta_0|)}{m+1}|\delta_0|<1 \quad (3.8)$$

where y_0 is a point between x_0 and x_*, so $g(x_0)\neq 0$

$$|(g(x_*))^{-1}g(x_0)|\geq 1-\frac{\varphi_0(|\delta_0|)}{(m+1)}|\delta_0|>0 \quad (3.9)$$

Hence, we get

$$|(g(x_0))^{-1}g(x_*)|\leq\frac{1}{1-\frac{\varphi_0(|\delta_0|)}{(m+1)}}=\frac{m+1}{m+1-\varphi_0(|\delta_0|)} \quad (3.10)$$

Using (2.2), (2.6), conditions (1.4), (1.5) and Lemma 2.3, we have

$$|(g(x_*))^{-1}g_0(x)|$$

$$|(g(x_*))^{-1}\int_0^1\ldots\int_0^1\theta_1^m\theta_2^{m-1}\ldots\theta_m f^{(m+1)}(x_*+\delta_0\theta_1\ldots\theta_m)d\theta_1\ldots d\theta_{m+1}$$

$$\leq\varphi_0^{(m)}(|\delta_0|) \quad (3.11)$$

And

$$|(g(x_*))^{-1}g_1(x_0)\delta_0|$$

$$|(g(x_*))^{-1}\int_0^1\ldots\int_0^1\theta_1^m\theta_2^{m-1}\ldots\theta_m$$

$$\times[f^{(m+1)}(x_*+\delta_0\theta_1\ldots\theta_{m-1})-f^{(m+1)}(x_*+\delta_0\theta_1\ldots\theta_m)]d\theta_1\ldots d\theta_{m+1}|$$

$$\leq\int_0^1\theta_1^m\theta_2^{m-1}\ldots\theta_m|(g(x_*))^{-1}[f^{(m+1)}(x_*+\delta_0\theta_1\ldots\theta_m)$$

$$-f^{(m+1)}(x_*+\delta_0\theta_1\ldots\theta_m)]|d\theta_1\ldots d\theta_{m+1} \quad (3.12)$$

$$\leq m!\int_0^1\theta_1^m\theta_2^{m-1}\ldots\theta_m\varphi(1-\theta_m)|\delta_0|)d\theta_1\ldots d\theta_{m+1}$$

$$:=\varphi^{(m)}(|\delta_0|)$$

In view of (3.10), (3.11) and (3.12), we obtain

$$|(g(x^*))^{-1}\alpha_0\leq(m+1)|(g(x^*))^{-1}g_0(x_0)(g(x_0))^{-1}g_0(x_0)||\delta_0|$$

$$+2\mathrm{m}|(\mathrm{g}(x^*))^{-1}\mathrm{g}_1(x_0)\delta_0|$$

$$\leq(\mathrm{m}+1)(\delta_0^{(m)}(|\delta_0|))^2\frac{(m+1)}{m+1-\varphi_0(|\delta_0|)|\delta_0|}|\delta_0| \quad (3.13)$$

$$+2m\varphi^{(m)}(|\delta_0|)$$

Since $|\delta_0|< r$, i.e., $h|\delta_0|> 0$, by (3.11)-(3.13), we get

$$|(g(x_*))^{-1}(\alpha_0\delta_0+2m\beta_0)|$$

$$=|(\mathrm{m}+1)(\mathrm{g}(x_*))^{-1}(\mathrm{g}(x_0))^{-1}g_0^2(x_0)\delta_0^2-2m(\mathrm{g}(x_*))^{-1}g_1(x_0)\delta_0^2$$

$$+2mg^{-1}(x_*)g_0(x_0)\delta_0+2m^2(\mathrm{g}(x_*))^{-1}(\mathrm{g}(x_0)|$$

$$\geq 2m^2-2m^2|g(x_*))^{-1}(\mathrm{g}(x_0)-g(x_*))|-2m|g(x_*))^{-1}(\mathrm{g}_0(x_0)\delta_0|$$

$$12m|(g(x_*))^{-1}g_1(x_0)\delta_0^2|-(m-1)|(g(x_*))^{-1}(\mathrm{g}_0(x_0)g(x_0))^{-1}(\mathrm{g}_0(x_0)\delta_0^2|$$

$$\geq 2m^2-\frac{2m^2\varphi_0(|\delta_0|)|\delta_0|}{m+1}-2m\varphi_0^{(m)}(|\delta_0|)|\delta_0|$$

$$-2m\varphi^{(m)}(|\delta_0|)$$

$$-(m+1)\varphi_0^{(m)}(|\delta_0|))^2\frac{m+1}{m+1-\psi_0(|\delta_0|)} \quad (3.14)$$

$$=pm(|\delta_0|)$$

$$\geq\frac{(m+1)^2\varphi_0^{(m)}(|\delta_0|)^2|\delta_0|^2}{m+1-\varphi_0(|\delta_j|)|\delta_0|}+2m\varphi^{(m)}(|\delta_0|)$$

We get by (3.8), (3.13) and (3.14)

$$|\delta_1|\leq\frac{|(g(x_*))^{-1}\alpha_0\delta_0^3|}{|(g(x_*))^{-1}(\alpha_0\delta_0^3+2\mathrm{m}\beta_j)|}$$

$$\leq\frac{g_1(|\delta_0|)|\delta_0|}{g_0(|\delta_0|)} \quad (3.15)$$

$$\leq c|\delta_0|<r$$

Where

$$c=\frac{g_1(|\delta_0|)}{g_0(|\delta_0|)}\in[0,1) \quad (3.16)$$

By simply replacing x_0, x_1 by x_k, x_{k+1} in the preceding estimates, we get

$$|x_{k+1}-x_*|\leq c|x_k-x_*|<r \quad (3.17)$$

so $\lim_{k\to+\infty}x_k=x_*$ and $x_{k+1}\in U(x_*,r)$

Next, we present a uniqueness result for the solution x_*

Proposition Suppose that the conditions (*A*) hold. Then, the limit point x_* is the only solution of equation $f(x)=0$ in $D_1=D\cap\overline{U}\ (x_*,\rho_0)$

Proof Let x_{**} be a solution of equation $f(x)=0$ in D_1: We can write by (2.8) that

$$f(x_{**})=g(x_{**})\,(x_{**}-x_*)^m \quad (3.18)$$

Using (1.4) and the properties of divided differences, we get in turn that

$$|1-g'(x_*)^{-1}g(x_{**})|=|\mathrm{g}(x_*)^{-1}(\mathrm{g}(x_*)-\mathrm{g}(x_{**})|$$

$$=|\mathrm{g}(x_*)-1\frac{f^{(m+1)}(z_0)}{(m+1)!}(x_{**}-x_*)| \quad (3.19)$$

$$\leq\frac{\varphi_0(|x_{**}-x_*|)|x_{**}-x_*)}{m+1}<1$$

for some point between x_{**} and x_* It follows from (3.18) and (3.19) $x_{**}=x_*$

Numerical Examples

We present a numerical example in this section.

Example Let $D=[0;1]$; $m=2$; $p=0$ and define function f on D by

$$f(x)=\frac{4}{35}x^{\frac{7}{2}}+\frac{1}{6}x^3+\frac{1}{2}x^2$$

We have $f'(x)=\frac{2}{5}x^{\frac{5}{2}}+\frac{x^2}{2}+x$, $f''(x)=x^{\frac{3}{2}}+x+1$, $f''(0)=1$ function f'' cannot satisfy (1.5) with ψ given by (1.6). Hence, the results in [4,5,12,13,16,17] cannot apply. However, the new results apply for $\varphi(t)=\frac{3}{2}t^{\frac{1}{2}}$ and $\varphi_0(t)=\frac{5}{2}$ Moreover, the convergence radius is r=0.8.

Example Let D=[-1, 1], m=2; p=0 and define function f on D by

$f(x)=e^x-x-1$

We get $\varphi_0(t)=\varphi(t)=et$. The convergence radius is r=1:4142; so choose r=1.

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