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Existence of Solutions for Impulsive Second Order Abstract Functional Neutral Differential Equation with Nonlocal Conditions and State Dependent-Delay

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Abstract

In this paper, we study the existence of mild solutions for the impulsive second order abstract partial neutral differential equations with state dependent delay of the form

$$\frac{d}{dt}[x'(t)+g(t,x_{\rho(t,x_t)})]=Ax(t)+f(t,x_{\rho(t,x_t)}),\quad t\in[0,a]\quad t\neq t_i$$

With nonlocal conditions

$$x(0)=x_0+p(x)\in B$$

$$x'(0)=y_0+q(x)\in X$$

$$\Delta x(t_i)=I_i^1 x(t_i),\ t=t_i\ \ i=1,2,...,n$$

$$\Delta x'(t_i)=I_i^2 x(t_i),\ t=t_i\ \ i=1,2,...,n$$

Keywords

Abstract Cauchy problem; Impulsive differential equations; Cosine function; State-dependent delay

Introduction

In this paper, we study the existence of mild solutions for the impulsive second order abstract partial neutral differential equations with state dependent delay of the form

$$\frac{d}{dt}[x'(t)+g(t,x_{\rho(t,x_t)})]=Ax(t)+f(t,x_{\rho(t,x_t)}),\quad t\in[0,a]\quad t\neq t_i \tag{1.1}$$

with the nonlocal conditions

$$x(0)=x_0+p(x)\in B \tag{1.2}$$

$$x'(0)=y_0+q(x)\in X \tag{1.3}$$

$$\Delta x(t_i)=I_i^1 x(t_i),\ t=t_i\ \ i=1,2,...,n \tag{1.4}$$

$$\Delta x'(t_i)=I_i^2 x(t_i),\ t=t_i\ \ i=1,2,...,n \tag{1.5}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operator $(C(t))_{t\in R}$ defined on a Banach space $(X,||.||)$, the function x_s: $(-\infty,a]\to X$, $x_s(\theta)=x(s+\theta)$ belongs to some abstract phase space B described axiomatically and f: Ix $B\to X$, g:Ix $B\to X$,ρ: BxX $(-\infty,a]$, p: $C(I;X)\to B$, q: $C(I;X)\to X$ and

I_i, J_i: $B\to X$, $i=1,2,\dots,n$ are appropriate functions and the symbol $\Delta\xi(t)$ represents the jump of the function ζ at t, which is defined by $\Delta\xi(t)=\xi(t^+)-\xi(t^-)$.

The theory of impulsive differential equations has become an important area of investigation in recent years stimulated by their numerous applications to problems arising in mechanics, electrical, engineering, medicine, biology, ecology etc [1-5].

Neutral functional differential equations with state- dependent delay and non-local conditions appear frequently in applications as model equations and for this reason the study of this type of equations has received great attention. The problem of the existence of solutions for second order functional differential equations with state-dependent delay and also nonlocal conditions have been treated in the literature recently in [6,7].To the best of our knowledge, the existence of solutions the impulsive second order abstract partial neutral functional differential equations with state-dependent delay and also nonlocal conditions is an untreated topic in the literature and this fact is the main motivation of the present work.

Preliminaries

Through this paper, A is the infinitesimal generator of strongly cosine function of bounded linear operators $(C(t))_{t\in R}$ on the Banach space $(X,||.||)$. we denote by $(S(t))_{t\in R}$ the associated sine function which is defined by $S(t)x=\int_0^t C(s)xds$, for $x\in X$ and tR In the sequel, N and $\bar{N}$ are positive constants such that $||C(t)||\leq N$ and $||S(t)||\leq \bar{N}$ for every $t\in I$.

In this paper $[D(A)]$ represents the domain of A endowed with the graph norm given by $||x||A=||x||+||ax||$,

$x\in D$ (A) while E stands for the space formed by the vectors $x\in X$ for which $C(.)x$ is of the class C^1 on R. We know from Kisinsky [8-10], that E endowed with the norm $||x||E=||x||+sup_{0\leq t<1}||AS(t)x||$, $x\in E$ is a Banach space. The operator-valued function

$A=\begin{bmatrix} C(t) & S(t)\\ AS(t) & C(t)\end{bmatrix}$ is a strongly continuous group of bounded linear operators on the space ExX generated by the operator $A=\begin{bmatrix} 0 & I\\ A & 0\end{bmatrix}$ defined on $D(A)$xE. It follows from this that $AS(t)$: $E\to X$ is a bounded linear operator and that $AS(t)x\to 0, t\to 0$, for each $x\in E$

Furthermore, if x: $(0,\infty]\to X$ is a locally integrable function, then $z(t)=\int S(t-s)x(s)ds$, defines an E-valued continuous function. This is a consequence of the fact that

$$\int_0^t G(t-s)\begin{bmatrix}0\\ x(s)\end{bmatrix}ds=\left[\int_0^t S(t-s)x(s)ds,\ \int_0^t C(t-s)x(s)ds\right]$$ defines an ExX- valued continuous function.

In this work we will employ an axiomatic definition for the phase space B. Specifically, B will be a linear space of functions mapping

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(-∞,0] into X endowed with a semi norm $\|.\|_B$ and satisfying the following assumptions:

(A1) If x: (-∞,b]→X, b>0, continuous on [0, b] and x_0B, then for every t[0, b] the following conditions hold:

(a) x_t is in B

(b) $\|x(t)\|\leq H\|x_t\|_B$

(c) $\|x_t\|B\leq M(t)\|x_0\|B+K(t)\sup\{\|x(s)\|: 0\leq s\leq t\}$we here $H > 0$ is a constant; K, M: [0, ∞)→{1, ∞), K is continuous, M is bounded and H,K,M are independent of x(.).

(A2) For the functions x in (A1), x_t is B valued continuous functions on [0,b].

(A3) The space B is complete.

Definition 2.1 (Mild solutions)

A function u: (-∞,a]→X is called a mild solution of the abstract Cauchy problem (1.1) – (1.3) for every $s\in I$ and

$$u(t)=C(t)(x_0+p(u))+S(t)(y_0+q(u)+g(0,x_{\rho(0,x_0)})$$
$$-\int_0^t C(t-s)g(s,x_{\rho(s,x_s)})ds+\int_0^t S(t-s)f(s,x_{\rho(s,x_s)})ds$$
$$+\sum_{0<t_i<t} C(t-t_i)I_i^1(x_{t_i})+\sum_{0<t_i<t} S(t-t_i)I_i^2(x_{t_i})$$

Some of our results is proved using the following well know results.

Theorem 2.2 (Leray Schauder Alternative)[4,pp,61]. Let D be a convex subset of a Banach space X and assume that 0∈D. Let G: D→D be a completely continuous map. Then the map G has a fixed point in D or the set $\{x\in D : x=\lambda G(x), 0<\lambda<1\}$ is unbounded [11-14].

Theorem 2.3 Sadovskii [15] Let D be a convex, closed and bounded subset of a Banach space X. If F: D→D is a condensing operator, then F has a fixed point in D.

Remark 2.4 The function t→ϕ_t is well defined and continuous from the set $R(\rho^-)=\rho\ (s,)$: $(s,\psi)\in IxB$, $(s,\psi)\leq 0$ in to B and there exists a continuous and bounded function J^ϕ: $R(\rho)\to(0,\infty)$ such that $\|\varphi_t\|_B\leq J\|\varphi_t\|^B$for every $t\in R(\rho)$.

Remark 2.5 The condition (2.4) is frequently verified by functions continuous and bounded. In fact, if B verifies axiom C_2 in the nomenclature of [12], then there exists $L < 0$ such that

$\|\varphi_t\|_B\leq L$ for every $\varphi\in B$ continuous and

bounded function. Consequently, $\|\varphi_t\|_B\leq L\dfrac{\sup_{\theta\leq 0}\|\varphi(\theta)\|}{\|\varphi\|_B}\|\varphi\|_B$ for every continuous and bounded function $\varphi\in B$ and every $t\leq 0$. We also observe that the space $C_r xL^p(g;X)$ verifies axiom C_2. In the rest of this paper, M_a and K_a are the constants defined by $M_a=\sup_{t\in J}M(t)$ and $K_a=\sup_{t\in J}K(t)$.

Using the following lemma for proof of our main result:

Lemma 2.6 [10,Lemma 2.1]

Let x:(-∞,a]→X be a function such that x_0=φ and $x_{[0,a]}\in PC$ Then $\|x_s\|_B\leq\{(M_a+\bar{J}^\varphi)\|\phi\|_B+K_a\sup\|x(\theta)\|;\theta\in[0,\max\{0,s\}]\}s\in R(\rho^-)$,

where $\bar{J}^\phi=\sup_{t\in R(\rho^-)}J^\phi(t)$.

The terminology and notations are those general used in functional analysis. In particular, for Banach spaces Z, W, the notation $L(Z,W)$ stands for the Banach space of bounded linear operators from Z into W and we abbreviate this notation to $L(Z)$ when $Z=W$. Moreover $B_r(x,Z)$ denotes the closed ball with radius $r > 0$ in Z and for a bounded function x:[0,a]→X and $0\leq t\leq a$ we employ the notation $\|x_t\|$ for $\|x_t\|=\sup\{\|x(s)\|: s\in[0,t]\}$

This paper has four sections. In the next section we establish the existence of mild solutions for the abstract Cauchy problem (1.1) - (1.3). In section 4 some applications are considered.

Existence of Solutions

In this section, we establish the existence of mild solution for the impulsive abstract Cauchy problem (1.1) – (1.5).

To prove our results, we assume that ρ: $IxB\to X$ is a continuous function and that the following conditions are verified.

(H1) The function f: $IxB\to X$ satisfies the following properties,

(a) The function $f(.,x):I\to X$ is strongly measurable for every $x\in B$.

(b) The function $f(t.,):B\to X$ is continuous for each $t\in I$.

(c) There exist an integrable function m_f: $I\to[0,\infty)$ and a continuous nondecreasing function W_f: $[0,\infty)\to(0,\infty)$ such that $\|f(t,x)\|B\leq m_f(t)W_f(\|x\|).)t,x)\in IxB$.

(H2) g: $IxB\to X$ is continuous function and verifies the following conditions:

(a) There exists a continuous function m_g: $[0,\infty)\to(0,\infty)$ and a continuous nondecreasing function W_g: $[0,\infty)\to(0,\infty)$ such that $\|g(t,x)\|_B\leq m_g(t)W_g(\|x\|),(t,x)\in IxB$

(H3) The maps I_j, J_i are continuous each function I_j is completely continuous and there are positive constants

$$\|I_i(\psi)\|\leq c_i^1\|\psi\|_B+c_i^2$$

$$\|J_i(\psi)\|\leq d_i^1\|\psi\|_B+d_i^2\quad i=1,2,...,n$$

(H4) There are positive constants P_i,Q_i such that

$$\|I_i(\psi_1)-I_i(\psi_2)\|\leq P_i\|\psi_1-\psi_2\|_B,\ \psi_1,\psi_2\in B,\ i=1,2,...,n$$

$$\|J_i(\psi_1)-J_i(\psi_2)\|\leq Q_i\|\psi_1-\psi_2\|_B,\ \psi_1,\psi_2\in B,\ i=1,2,...,n$$

Theorem 3.1 Assume that the conditions (H1) – (H3) are verified and that $g(.)$ is completely continuous. Suppose, furthermore that the following conditions hold:

(a) for every $0<t'<t$ and $r > 0$, the set $U(t,t',r)=\{S(t')f(S,x)$: $s\in\{0,t],\|x\|\leq r^*$ is compact in X.

(b) $p(.)$ is completely continuous and there is N_p>0 such that $\|p(u)\|\leq N_p$ for every $u\in C(I;X)$

(c) for every $s\in I$ and every $r > 0$ the set $V(s,r)=\{S(s)q(x): \|x\|\leq r^*$ is relatively compact in X and there is N_q>0 such that $\|q(u)\|\leq N_q$ for every $u\in C(I;X)$

If $\mu=1-K_a[N\sum_{i=1}^n c_i^1+\bar{N}\sum_{i=1}^n d_i^1]$ and

$$\int_{\beta_\lambda(0)=C}^{\beta_\lambda(t)}\frac{ds}{W_g(s)+W_f(s)}\leq\frac{K_a}{1-\mu}\int_0^a(Nm_g(s)+\bar{N}m_f(s))ds<\int_C^\infty\frac{ds}{W_g(s)+W_f(s)}$$

where

$$C=\frac{1}{1-\mu}\Big(K_a(NH\|x_0\|+NHl_p r+M_a+\bar{J}^{\varphi})\|\varphi\|_b+K_a(\bar{N}\|Y_0\|+\bar{N}l_q$$
$$+\bar{N}m_g(s)W_g((M_a+\bar{J}^{\varphi})\|\varphi\|_B+K_a\|u^{\lambda}\|_a))$$
$$+K_a\sum_{i=1}^{n}\left[Nc_i^2+Nc_i^1\xi^{\lambda}(t)+\bar{N}d_i^2+\bar{N}d_i^1\xi^{\lambda}(t)\right]$$
$$+K_a\left[N\sum_{i=1}^{n}c_i^2+\bar{N}\sum_{i=1}^{n}nd_i^2\right]\Big)$$

Then there exists a mild solution of (1.1) – (1.5).

Proof

On the space $C(I;X)$ we define the map Γ: $C(I;X)\to C(I;X)$ by

$$\Gamma u(t)=C(t)\big(x_0+p(u)\big)+S(t)\big(y_0+q(u)+g(0,x_{\rho(0,x_0)})\big)$$
$$-\int_0^t C(t-s)g(s,x_{\rho(s,x_s)})ds+\int_0^t S(t-s)f(s,x_{\rho(s,x_s)})ds$$
$$+\sum_{0<t_i<t}C(t-t_i)I_i^1(x_{t_i})+\sum_{0<t_i<t}S(t-t_i)I_i^2(x_{t_i})$$

In order to use Leray Schauder alternative and from assumption (A1).

We obtain an a priori bounded for the solution of the integral equation $u=\lambda\Gamma(u)$, $\lambda\in(0,1)$ if u^{λ} is a solution of $u=\lambda\Gamma(u)$, $\lambda\in(0,1)$

we get,

$$\|u^{\lambda}(t)\|\le N(H\|x_0\|+HNl_p)+\bar{N}\|y_0\|+\bar{N}l_q+m_gW_g((M_a+\bar{J}^{\varphi})\|\varphi\|_B+K_a\|u^{\lambda}\|_a)$$
$$+N\int_0^t m_g(s)W_g(M_a+\bar{J}^{\varphi}\|\varphi\|_B+K_a\|u^{\lambda}\|_a)ds$$
$$+\bar{N}\int_0^t m_f(s)W_f(M_a+\bar{J}^{\varphi}\|\varphi\|_B+K_a\|u^{\lambda}\|_a)ds$$
$$+N\sum_{0<t_i<t}c_i^1\Big((M_a+J_0^{\varphi})\|\varphi\|_B+K_a\sup_{\theta\in[0,t]}\|u^{\lambda}(\theta)\|\Big)+N\sum_{i=1}^{n}c_i^2$$
$$+\bar{N}\sum_{0<t_i<t}d_i^1\Big((M_a+J_0^{\varphi})\|\varphi\|_B+K_a\sup_{\theta\in[0,t]}\|u^{\lambda}(\theta)\|\Big)+\bar{N}\sum_{i=1}^{n}d_i^2$$

Denoting by the $\beta_{\lambda}(t)$ right hand of above equation follows that,

$$\beta_{\lambda}'(t)\le\frac{K_a}{1-\mu}(Nm_g(t)+\bar{N}m_f(t))(W_g(\beta_{\lambda}(t)+W_f(\beta_{\lambda}(t))$$

and hence,

$$\int_{\beta_{\lambda}(0)=C}^{\beta_{\lambda}(t)}\frac{ds}{W_g(s)+W_f(s)}\le\frac{K_a}{1-\mu}\int_0^a(Nm_g(s)+\bar{N}m_f(s))ds<\int_C^{\infty}\frac{ds}{W_g(s)+W_f(s)}.$$

Which implies that the set of function $\{\beta_{\lambda}(.):\lambda\in(0,1)\}$ is bounded in $C(I;R)$. This prove that $\{U^{\lambda}(.):\lambda\in(0,1)\}$ is also bounded in $C(I;X)$.

Next, we prove that $\Gamma$is completely continuous. To this end, we introduce the decomposition $\Gamma=\Gamma_1+\Gamma_2+\Gamma_3$ where,

$\Gamma_1u(t)=C(t)(x_0+p(u))+S(t)(Y0+q(u)+g(0,x_{\rho(0,x0)})$

$$\Gamma_2u(t)=-\int_0^t C(t-s)g(s,x_{\rho(s,x_s)})ds+\int_0^t S(t-s)f(s,x_{\rho(s,x_s)})ds$$

$$\Gamma_3=\sum_{0<t_i<t}C(t-t_i)I_i^1(x_{t_i})+\sum_{0<t_i<t}S(t-t_i)I_i^2(x_{t_i})$$

It is easy to show that Γ_1 is completely continuous and that Γ_2 is continuous. Next, by using Ascoli Arezela we prove that $\Gamma(B_r(0,C(I;X)))$ is relatively compact $C(I;X)$. In the sequel $B_r=B_r(0,C(I;X))$

Step:1

The set $\Gamma_2(B_r=\{\Gamma_2u:u\in B_r\}$ is equicontinuous on I. Let $t\in I$ and $g(.)$ is completely continuous, there exist $\delta>0$ such that

$||C(s+h)-C(s)g(s',x)||\le\varepsilon$, $||x||\le r^*$ where $r^*=((M_a+\bar{J}^{\varphi})\|\varphi\|_B+K_a(r),(s,s')\in I^2$ when $|h|\le\delta$. For $u\in B_r$ and $|h|\le\delta$ with $t+h\in I$, we get

$$\|\Gamma_2u(t+h)-\Gamma_2h(t)\|\le\int_0^t\|C(t+h-s)-C(t-s)g(s,x_{\rho(s,x_s)})\|ds+N\int_t^{t+h}\|g(s,x_{\rho(s,x_s)})\|ds$$
$$+\int_0^t\|s(t+h-s)-s(t-s)f(s,x_{\rho(s,x_s)})\|ds+\bar{N}\int_t^{t+h}\|f(s,x_{\rho(s,x_s)})\|ds$$
$$\le\varepsilon t+NW_g(r*)\int_t^{t+h}m_g(s)ds+\bar{N}hW_f(r*)\int_0^t m_fds+\bar{N}W_f(r*)\int_t^{t+h}m_fds$$

which prove the assertion.

Step:2

The set $\Gamma_2(B_r)(t)=\{\Gamma_2u(t):u\in B_r\}$ is relatively compact in X for every $t\in I$ Let $t\in I$ and $\varepsilon>0$ If $u\in B_r$, $x\in B_r$ from the estimate, $||f(\theta,u(\theta))||\le m_f(\theta)w_f(||u(\theta)||)\le m_f(\theta)w_f(r^*)$ follows that the set $U=\{f(t-s),x(t-s);s\in\{0,t\},u\in B_r\}$is bounded in X. Using that $S:I\to L(X)$ is uniformly Lipschitz on I, we can chose $0=S_1<S_2<\ldots<S_k=t$ such that $||S(s')y-S(s)y||<\varepsilon$, $y\in U$, where $S,s'\in[S_i,S_{i+1}]$ for some $i=1,2,\ldots,k-1$. Let $x\in B_r$ Bocher integral see [13,lemma 2.1.3] and fact that $V=\{C(s)g(s',x)\}$ is relatively compact in X, follows that,

$$\Gamma_2u(t)=-\int_0^t C(t-s)g(s,(x)_{\rho(t-s,(x)_{t-s})})ds+\sum_{i=1}^{n-1}\int_{s_i}^{s_{i+1}}(S(s)-S(s_i))f(t-s,(x)_{\rho(t-s,(x)_{t-s})})ds$$
$$+\sum_{i=1}^{n-1}\int_{s_i}^{s_{i+1}}S(s_i)f(t-s,(x)_{\rho(t-s,(x)_{t-s})})ds$$
$$\in tco(v)+B_a(0,X)+\sum_{i=1}^{n-1}(s_{i+1}-s_i)Co(U(t,s_i,r))$$

where $co(Q)$ denote the convex hull of a set Q. Thus $\Gamma_2(B_r)(t)$ is relatively compact in X. From the steps 1 and 2, follows that $\Gamma_2(B_r)$ is relatively compact in $C(I;X)$ and so that $\Gamma_2$is completely continuous. Finally, the theorem 1.1 assert that Γ has a fixed in $C(I;X)$. The proof is complete.

If the maps g,p,q fulfill some Lipschitz conditions instead of the compactness properties considered in the preceding theorem, we also can establish a result of existence.

Theorem 3.2

Assume that (H1) and (H4) are verified and that the following conditions hold;

(a) for every $0<t'<t$ and $r>0$, the set $U(t,t',r)=\{S(t')f(s,x): s\in[0,t]\}$ is relatively compact in X.

(b) There exists positive constants l_g,l_p and l_q such that,

$$\|g(t,x_1)-g(t,x_2)\|\le l_g\|x_1-x_2\|,\quad (t,x_i)\in I\times X$$
$$\|p(u)-p(v)\|\le Hl_p\|u-v\|,\quad u,v\in C(I;X)$$
$$\|q(u)-q(v)\|\le l_q\|u_v\|,\quad u,v\in C(I;X)$$

and

$$N(Hl_p+l_ga)+\bar{N}(l_q+l_g)+K_a\bar{N}\liminf_{\xi\to\infty}\frac{W_f(\xi)}{\xi}\int_0^a m_f(s)ds+K_a\sum_{i=1}^{n}[NP_i+\bar{N}Q_i]<1 \qquad (3.1)$$

Then there exists a mild solution of (1.1) – (1.5).

Proof

Let $Y=C(I;X)$ and $\Gamma=\Gamma_1+\Gamma_2+\Gamma_3$: $Y\to Y$ be the map defined by

$$\Gamma_1 u(t)=C(t)(x_0+p(u))+S(t)(y_0+q(u)+g(0,x_{\rho(0,x_0)})-\int_0^t C(t-s)g(s,x_{\rho(s,x_s)})ds$$

$$\Gamma_2=\int_0^t S(t-s)f(s,x_{\rho(s,x_s)})ds$$

$$\Gamma_3=\sum_{0<t_i<t} C(t-t_i)I_i^1(x_{t_i})+\sum_{0<t_i<t} S(t-t_i)I_i^2(x_{t_i})$$

We affirm that there exists $r>0$ such that $\Gamma(B_r(0,Y))\subset B_r(0,Y)$ In fact, if we assume the affirmation is false, then for each $r>0$ there exists $u^r\in B_r(0,Y)$ such that $||\Gamma u^r||>r$. which imply that

$$r\le\|\Gamma u^r\|\le N(H\|x_0\|+Hl_p r+\|p(0)\|)+\bar{N}(\|y_0\|+l_q r+\|q(0)\|+l_g r+\|g(0,0)\|)$$
$$+N\int_0^a (l_g\|u^r(s)\|+\|g(s,0)\|)ds$$
$$+\bar{N}\int_0^a m_f(s)W_f(\|u^r(s)\|)ds$$

and so that

$$1\le N(Hl_p+l_g a)+\bar{N}(l_q+l_g)+K_a\bar{N}\liminf_{\xi\to\infty}\frac{W_f(\xi)}{\xi}$$
$$\int_0^a m_f(s)ds+K_a\sum_{i=1}^n[NP_i+\bar{N}Q_i]$$

which is an absurd.

Step:1

Let $r_0>0$ such that $\Gamma(B_r(0,Y))\subset B_{r0}(0,Y)$ using the steps in the proof of theorem (3.1), follows that Γ_2 is completely continuous and from the estimate

$$\|\Gamma_1 u-\Gamma_1 v\|\le(N(Hl_p+l_g a)+\bar{N}(l_q+l_g))\|u-v\|$$

such that Γ_1 is a contraction.

Step:2

The map Γ_3 is a contraction on $B_r(0,Y)$. The assertion follows directly from (3.1) and the estimate,

$$\|\Gamma_3 x-\Gamma_3 y\|\le\sum_{i=1}^n[NP_i+\bar{N}Q_i]\|u-v\|_{PC}$$

Thus, Γ is a condensing map on $B_{r_0}(0,Y)$. The assertion is now consequence of the Sadovskii's point theorem, see [15,16].

The proof is finished.

Conclusion

In this section we consider the applications of our abstract result.

We discuss the existence of solutions for the partial differential system with state-dependent delay and nonlocal conditions:

$$\frac{\partial^2 u(t,\xi)}{\partial t^2}+\int_{-\infty}^t a_1(s-t)u(s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta,\xi\right)ds=$$
$$\frac{\partial^2 u(t,\xi)}{\partial t^2}+\int_{-\infty}^t b_1(s-t)u(s-\rho_1(t)\rho_2\left(\int_0^\pi b_2(\theta)|u(t,\theta)|^2 d\theta,\xi\right)ds \tag{4.1}$$

for $t\in I=[0,a],\xi\in[0,\pi]$, subject to the nonlocal conditions

$$u(0,\xi)=x_0+\sum_{i=1}^n\alpha_i u(t_i),\quad \xi\in J \tag{4.2}$$

$$\frac{\partial u(0,\xi)}{\partial t}=y_0+\sum_{i=1}^n\beta_i u(t_i),\quad \xi\in J \tag{4.3}$$

$$\Delta u(t_i)(\xi)=\int_{-\infty}^{t_i} c_i(t_i-s)u(s,\xi)ds \tag{4.4}$$

$$\Delta u'(t_i)(\xi)=\int_{-\infty}^{t_i}\bar{c}_i(t_i-s)u(s,\xi)ds \tag{4.5}$$

where $0<t_i,s_j<a,\alpha_i\in B$ and $\beta\in R$ are fixed numbers $x_0\in B$, $Y_0\in X$. By the definition of the functions $p(u)\xi=\sum_{i=1}^n\alpha_i u(t_i)$ and $q(u)\xi=\sum_{i=1}^n\beta_i u(t_i)$. The system (4.1) – (4.3) can be described as the abstract Cauchy problem with state-dependent delay and nonlocal conditions. To apply our abstract results, we consider the space $X=L^2([0,\pi]);B=C_0xL^2(g,x)$ and the operator $Af=f''$ with domain

$D(A)=\{x\in X$: $x''\in X$, $x(0)=x(\pi)=0\}$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in R}$ on X. Furthermore, A has a discrete spectrum, the eigen value are $-n^2$, $n\in N$, with corresponding eigen vectors the following properties hold

(a) The set $\{Z_n: n\in N\}$is an orthonormal basis of X.

(b) For $x,y\in X, C(t)x=\sum_{n=1}^\infty\cos(nt)(x,z_n)_{z_n};\|C(t)\|=\|s(t)\|\le 1$ for all $t\in R$ and that $S(t)$ is compact for every $t\in R$

(c) If Φ is the group of translations on X defined by $\Phi(t)x(\xi)=\bar{x}(\xi+t)$, where $\bar{x}$ is the extension x with period 2π, then $C(t)=\frac{1}{2}(\Phi(t)+\Phi(-t))$ and $E=\{x,y\in H^1(0,\pi)$: $x(0)=x(\pi)=0$ see [4] for details.

(d) The function $c_i\in C([0,\infty);R)$ and $P_i^1=\left(\int_{-\infty}^0\frac{(c_i(s))^2}{h(s)}ds\right)\frac{1}{2}<\infty$

(e) The function $\bar{c}_i\in C([0,\infty);R)$ and $Q_i^1=\left(\int_{-\infty}^0\frac{(\bar{c}_i(s))^2}{h(s)}ds\right)\frac{1}{2}<\infty$

Assume that $\varphi\in B$ the functions a_i: $R\to R$, b_i: $R\to R$, and pi: $[0,\infty)\to[0,\infty)$, $i=1,2$ are continuous, $a_2(t)\ge 0$ and $b_2(t)\ge 0$ for all $t\ge 0$ and

$$L_1=\int_0^\infty\left(\frac{a_1^2(s)}{g(s)}\right)^{\frac{1}{2}}ds<\infty, L_2=\int_0^\infty\left(\frac{b_1^2(s)}{f(s)}\right)^{\frac{1}{2}}ds<\infty$$

Under these conditions we can define the operators f: $IxB\to X$; g: $B\to X$ and I_i,J_i:$B\to X$ and ρ: $IxB\to R$ by

$$f(t,x)(\xi)=\int_{-\infty}^0 b_1(s)x(s,\xi)ds$$

$$g(t,x)(\xi)=\int_{-\infty}^0 a_1(s)x(s,\xi)ds$$

$$I_i^1(\psi)(\xi)=\int_{-\infty}^0 c_i(s)\psi(s,\xi)ds$$

$$I_i^2(\psi)(\xi)=\int_{-\infty}^0\bar{c}_i(s)\psi(s,\xi)ds$$

and transform system (4.1) – (4.5) in to the abstract Cauchy problem (1.1) – (1.5). Moreover f is a continuous linear operator with $||f||\le L_1,||g||\le L_2,\rho$, is continuous and $\rho(t,\psi)\le s$ for every $S\in[0,a]$. $||f(t,\psi)||\le d_1(t)+d_2(t)||\psi||_B$ for every $t\in[0,a]$ where

$$d_1(t) = \left(\int_0^{\pi} \left(\int_{-\infty}^{0} v(t,s,\xi) ds \right)^2 d\xi \right)^{\frac{1}{2}} \text{ and}$$

$$d_2(t) = \left(\int_{-\infty}^{0} \frac{\mu(t,s)^2}{h(s)} ds \right)^{\frac{1}{2}}$$

Case(i) Assume that φ satisfies (Remark 2.4). Then there exists a mild solution of (4.1) – (4.5).

Case(ii) If φ is continuous and bounded, then there exists a mild solution of (4.1) – (4.5)

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