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Formal Power Series of Logarithms

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Abstract

Since its beginning with Barrow and Newton, calculus of finite differences has been viewed-whether admittedly or not-as a proper method of computation with special functions. Until the nineteenth century, when the trials of convergence were to be gradually imposed, mathematicians handling difference equations and series involving polynomials and logarithms weren't beset by doubts on the correctness of their manipulations, and actually, their results have seldom turned out to be incorrect, even by the standards of our day.

There were, however, some embarrassing exceptions-which mathematicians during this century have chosen largely to ignore. Perhaps the simplest known of those is that the Euler-MacLaurin summation formula. When applied to any "function" aside from a polynomial, this formula gives a divergent series, which nonetheless are often wont to obtain astonishingly good numerical approximations, and which may be used without worrying in formal manipulations. It's of little help to justify such manipulations by appealing to Poincarit's definition of an asymptotic expansion. Most Euler-MacLaurin series contain logarithmic terms and other functions growing slower than any polynomial, whose asymptotic expansion consistent with Poincart would equal zero. The suggestion first made by Dubois-Reymond, and later haunted by G. H. Hardy-that the notion of an asymptotic expansion be reinforced by logarithmic scales-has not been developed, neither is it clear how the difficulties of its implementation are to be surmounted, or maybe whether such difficulties are worth surmounting.

Browbeaten by demands of rigor that might eventually be seen as spurious, the first difference-equationists of this century-such stalwarts as Milne-Thompson, NGrlund, Pincherle, and Steffensen-went to great lengths to plan acceptable definitions of the "natural" solutions of difference equations. With due reference to our predecessors, we submit that their proposals can nowadays be classified as pointless. What's needed instead so as to justify formal manipulations-we can state nowadays with the confidence that comes after fifty years of local algebra-is an extension of the notion of formal series which will include, besides powers of x, powers series in other special functions of classical analysis, most notably exponentials and logarithms. It's the aim of this work to hold out such an extension, one that has the formal theory of infinite series in polynomials and logarithms.

The difficulty of this program, and maybe the rationale why it's not been previously administered, is that the algebraic unwieldiness of the functions x"(log x)', where n is an integer and t a nonnegative integer. The coefficients within the Taylor expansions in powers of a of the functions (x + a)" (log(x + a))' don't seem to follow much rhyme or reason, and unless these coefficients are somehow made easy to handle, no simple operational calculus are often obtained. We resolve this difficulty by introducing another basis for the vector space spanned by the functions x"(log x)', a basis whose members we call the harmonic logarithms. We denote by i(x) the harmonic logarithm of order t and degree n. For positive integers t and integers n, or for t=0 and nonnegative integers n, the harmonic logarithms of order i and n span the subspace L of the logarithmic algebra spanned by x"(log x)' over an equivalent values of n and t. especially , L'" is that the subspace of ordinary polynomials in x. When t may be a positive integer, however, the variable n is allowed to vary over all integers, positive or negative. One has, for instance, for t=1.

The harmonic logarithms end up to satisfy an identity which "logarithmically" generalizes the theorem for polynomials, to with the coefficients are generalizations of the binomial coefficients (and, in fact, coincide with the binomial coefficients altogether cases during which the binomial coefficients are defined). We propose to call them the Roman coefficients. They satisfy identities almost like those satisfied by the binomial coefficients, and that they are defined for all integers n and k.

The ring of formal series of logarithmic type is now defined in two steps in terms of the harmonic logarithms. First, one completes the subspace L into a vector space LY in such how as to get formal series of the shape and second, one takes the algebraic union 9 of the vector spaces LZ, thereby obtaining the logarithmic algebra. Thus, a proper series of logarithmic type may be a finite sum of (infinite) formal series of the shape (2) for various values of t. Note that for t=0, one obtains nothing quite polynomials in x.

The ring of formal differential operators with constant coefficients acts on the logarithmic algebra 9. More pleasingly, the ring of all formal Laurent series within the derivative D acts on the subspace Y which is that the union of all subspaces 9 (') for t positive. The subspace 2 seems to be the "largest" subspace of the logarithmic algebra on which the derivative operator is invertible. This fact (together with certain commutation relations explained within the text) results in a definition of natural solutions of difference equations which is more general than those previously given by Milne-Thompson and Norlund, and which coincides with them altogether cases where both are defined.

The rest of this work develops logarithmic analogs of variednotions that were previously only known for polynomials, notably the logarithmic analog of Appell polynomials and therefore the logarithmic analog of the idea of sequences of binomial type, which we call Roman graded sequences. Several special cases are figured out , notably the logarithmic analogs of Bernoulli and Hermite polynomials, of Gould and Laguerre polynomials, and of the factorial powers of calculus of finite differences. Within the last example, one finds that the Gauss $-function (the logarithmic derivative of the gamma function) is one term within the logarithmic extension of the lower factorial function. Thus, classical identities satisfied by the $-function are seen to be trivial consequences of general logarithmic identities. We stress the very fact that the examples given here are only a sampling of the special functions which will be brought under the logarithmic umbrella.

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