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Meromorphic Functions and Theta Functions on Riemann Surfaces

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Abstract

Theta functions play a major role in many current researches and are powerful tools for studying integrable systems. The purpose of this paper is to provide a short and quick exposition of some important aspects of meromorphic theta functions for compact Riemann surfaces. The study of theta functions will be done via an analytical approach using meromorphic functions in the framework of Mumford. Some interesting examples will be given: the classical Kirchhoff equations in the cases of Clebsch and Lyapunov-Steklov, the Landau-Lifshitz equation and the sine-Gordon equation.

Keywords

Riemann surfaces; Meromorphic functions; Theta functions

Introduction

Let X be a compact Riemann surface of genus $g \geq 1$ and $(b_{jk})_{1\leq j,k\leq g}$ a square matrix of order g, symmetric and Im $B > 0$. We consider the Riemann theta function $\theta(z|B)$ defined by its Fourier series:

$$\sum_{m\in Z^g} e^{\pi i\langle Bm,m\rangle + 2\pi i\langle z,m\rangle}, z \in C^g \quad (1)$$

Where

$$\langle z,m\rangle = \sum_{j=1} Z_j m_j.$$

$$\langle z,m\rangle = \sum_{j=1}^{g} Z_j m_j.$$

The convergence of this series for all $z \in C^g$ results from the fact that Im $B > 0$. We show that this series converges absolutely and uniformly on compact sets and thus the function $\theta(z|B)$ is holomorphic over C^g.

When the matrix B is fixed, we will put in the sequel $\theta(z) \equiv \theta(z|B)$. Let $(e_1, ..., e_g)$ be a basis of $\mathbb{C}^g$ with $(e_j)_k = \delta_{jk}$ and let $f_j = (b_{1j} ... b_{gj})^T$ be the columns of the matrix B or in condensed form $f_j = b_{ej}$, $j = 1, ..., g$.

Theorem 1

The function θ satisfies the functional equations

$$\theta(z + e_j) = \theta(z)$$

$$\theta(z + f_j) = \; = e^{-\pi i b_{jj} - 2\pi i z_j}\theta(z), \quad (2)$$

For any $m, n\ Z^g$, we have

$$\theta\,(z + n + B_m) = e^{-\pi i\langle Bm,m\rangle - 2\pi i\langle m,z\rangle}\theta(z) \quad (3)$$

Any vectors of the form $n + B_m$ is a period of the Riemann theta function, they constitute the period lattice.

Proof: The first relation results from formula (1). Concerning the second relation, we have

$$\theta(z + f_j) = \sum_{m\in Z^g} e^{\pi i\langle Bm,m\rangle + 2\pi i\langle m,z+f_i\rangle}$$

$$= \sum_{m\in Z^g} e^{\pi i\langle B(n-e_j),n-e_j\rangle + 2\pi i\langle (n-e_j),z+f_i\rangle}, n \equiv m + e_j$$

$$= e^{\pi i\langle Be_j,e_j\rangle - 2\pi i\langle e_j,z\rangle}\theta(z)z$$

$$= e^{-\pi i b_{jj} - 2\pi i z_j}\theta(z),$$

And, the relation (3) immediately follows.

Thus the vectors $e_1, ..., e_g$ are the periods of function $\theta(z)$. The vectors $f_1, ..., f_g$ are called the quasi-periods. The function θ is quasi-periodic and is well defined on the Jacobian variety of X.

Consider a generalization of the theta function (1) called the theta function with characteristics α and β

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z\,|\,B)\ \sum_{m\in Z^g} e^{\pi i\langle B(m+\alpha),m+\alpha\rangle + 2\pi i\langle z+\beta),m+\alpha\rangle},\ \ \alpha,\beta \in R^g \quad (4)$$

$$= e^{\pi i\langle B\alpha,\alpha\rangle + 2\pi i\langle z+\beta,\alpha\rangle}\theta(z + \beta + B\alpha) \quad (5)$$

To simplify the formulas, we simply note:

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z) \equiv \theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z\,|\,B)$$

when the matrix B is fixed. In particular,

$$\theta\begin{bmatrix}0\\ 0\end{bmatrix}(z) + \theta(z)$$

From equation (3), we have also

$$\theta\begin{bmatrix}m\\ n\end{bmatrix}(z) + \theta(z),\ \ m,n \in Z^g$$

Consequently, it is sufficient to consider the functions $\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z)$ where $\alpha = (\alpha_1, \ldots \alpha_g)$, $\beta = (\beta_1, \ldots, \beta_g) \in R^g$ are such that: $0 < \alpha_j, \beta_j < 1$, $j = 1, ..., g$.

Theorem 2

The periodicity property of the theta-functions with characteristics is given by the following relation.

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z)(z + n + Bm) = e^{-\pi i\langle Bm,m\rangle - 2\pi i\langle z,m\rangle + 2\pi i\langle \alpha,n\rangle(\langle \alpha,n\rangle - \langle \beta,m\rangle)}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z)$$

Proof

It suffices to reason as in the previous proposition.

If $\alpha_1, \ldots \alpha_g$ and $\beta_1, \ldots, \beta_g$ a are 0 or , we will say that the set $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is a

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half period. In addition, a half period $\begin{bmatrix}\alpha\\\beta\end{bmatrix}$ is said to be even if 4 $(\alpha, \beta) \equiv 0$ (mod. 2) and odd if not.

Theorem 3

The function $\theta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(z)$ is even if half-period *(z)* is even and odd if half-period $\begin{bmatrix}\alpha\\\beta\end{bmatrix}$ is odd. In addition, we have $\theta(z)=\theta(-z)$.

Proof: By making the substitution

$z \to -z,\ m \to -m-2,$

into (4), we obtain immediately for the general term of the series,

$$e^{\pi i\langle B(-m-\alpha)-m-\alpha\rangle,+2\pi i\langle -z+\beta,-m-\alpha\rangle} =$$

$$e^{\pi i\langle B(m+\alpha),m+\alpha\rangle,+2\pi i\langle z+\beta,m+\alpha\rangle} e^{4\pi i\langle \alpha,\beta\rangle}$$

Now, from the above definition, the sign $e^{4\pi i(\alpha,\beta)}$ is determined by the parity of the number 4 (α, β), and the last relation results.

For example, the number of even half-periods is equal to $2^{g-1}(2^g+1)$ and of odd half-periods to $2^{g-1}(2^g-1)$.

Meromorphic Functions Expressed in Terms of Theta Functions

Consider the case of Riemann surfaces of genus 1, i.e., elliptic curves. Let us recall that an elliptic function is a doubly periodic meromorphic function. In this case, the matrix *B* is reduced to a number that we denote by b with Im b≥0. The numbers 1 and b generate a parallelogram of the periods denoted Ω. The four theta functions corresponding to the half-periods $\begin{bmatrix}1/2\\1/2\end{bmatrix},\begin{bmatrix}1/2\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\1/2\end{bmatrix}$ are.

$$i\theta_1(z)\equiv\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z),$$

$$\theta_2(z)\equiv\theta\begin{bmatrix}1/2\\0\end{bmatrix}(z),$$

$$\theta_3(z)\equiv\theta\begin{bmatrix}0\\0\end{bmatrix}(z)=\theta(z)$$

$$\theta_4(z)\equiv\theta\begin{bmatrix}0\\1/2\end{bmatrix}(z).$$

These functions are holomorphic on. Moreover, we immediately deduce from theorem 3 that $\theta_1(z)$ is odd and that $\theta_2(z)$, $\theta_3(z)$, $\theta_4(z)$, are even. To determine the zeros of the functions θ_j it is sufficient, from theorem 2, to look for them in the parallelogram of the periods Ω. Since $\theta_1(z)$ is odd, then $\theta_1(0)=0$ and the other zeros of $\theta_j(z)$ are obtained via theorem 2. In particular we see that $\theta_3\left(\frac{1}{2}+\frac{b}{2}\right)=0$. We claim that this zero of $\theta_3(z)$, in

Ω is unique, which is easy to prove. Namely, we need to prove that along boundary δΩ.

$$\frac{1}{2\pi i}\int_{\partial\Omega} d\log\theta_3(z)=1$$

We have

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{d\theta_3(z)}{\theta_3(z)}=\frac{1}{2\pi i}\int_0^1(d\log\theta_3(z)-d\log\theta_3(z+b))$$

$$+\frac{1}{2\pi i}\int_0^b(d\log\theta_3(z+1)-d\log\theta_3(z))$$

From theorem 2, we have $\theta_3(z+1)+\theta_3(z)\, and\, \theta_3(z+b)=e^{-\pi ib-2\pi iz}\theta_3(z),$

So

$$\int d\log\theta_3(z+1)-d\log\theta_3(z))=0,$$

And

$$\int_0^b(d\log\theta_3(z)-d\log\theta_3(z+b))=\int_0^b(d\log\theta_3(z)-d\log e^{-\pi ib-2\pi iz}\theta_3(z)),$$

$$=\int_0^b 2\pi i dz=1,$$

Therefore

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{d\theta_3(z)}{\theta_3(z)}=\frac{1}{2\pi i}\int_{\partial\Omega} d\,log\theta_3(z)=1,$$

and we have the following result:

Theorem 4

The function *θ(z)* has in the parallelogram of the periods Ω generated by 1 and b), only one zero at the point z= $\frac{1}{2}$ (1 + b).

By putting z = x ∈ R, b = it, t ∈ R_+, we shall see (following [16]) θ (x|it) as the fondamental periodic solution to the heat equation. Equation (1) is written

$$\theta(x\,|\,it)=\sum_{m\in Z}e^{-\pi m^2t+2\pi imx}=1+2\sum_{m=1}^{\infty}e^{-\pi m^2t}\cos 2\pi mx$$

Thus θ is a real valued function of two real variables. This function is periodic with respect to x, i.e., θ(x+1|it)=θ (x|it). Since

$$\frac{\partial\theta(x\setminus it)}{\partial t}=2\sum_{m=1}^{\infty}(-\pi m^2)e^{-\pi m^2t}\cos\pi mx,$$

$$\frac{\partial\theta(x\setminus it)}{\partial x^2}=2\sum_{m=1}^{\infty}(-4\pi^2m^2)e^{-\pi m^2t}\cos\pi mx,$$

Then

$$4\pi\frac{\partial\theta(x\setminus it)}{\partial t}=\frac{\partial^2\theta(x\setminus it)}{\partial x^2}$$

This suggests that we characterise the theta function *θ(x|it)* as the unique solution to the heat equation with a certain periodic initial data when t=0. To examine the limiting behaviour of f, we integrate it against a test periodic function

$$f(x)=\sum_n a_n e^{2\pi inx}$$

Then

$$\lim_{t\to 0}\int_0^1\theta(x\,|\,it)f(x)dx=\int_0^1\sum_{m,n}a_n e^{-\pi m^2t+2\pi i(m+n)x}dx,$$

$$=\lim_{t\to 0}\sum_{m,n}a_n e^{-\pi m^2t}\int_0^1 e^{-2\pi mi(m+n)x}dx,$$

$$=\lim_{t\to 0}\sum_m a_m e^{-\pi m^2t}$$

$$=\sum_m a_m$$

=f(0)

Hence *θ(x|it)* converges, as a distribution, to the sum of the delta functions at all integral points x∈ Z as t→0. The uniqueness of this solution results from the fact that

$$\lim_{t\to 0}\theta(x\mid it)=\sum_{m=-\infty}^{\infty}\delta_m(x),$$

Where δ_m is the distribution of Dirac at m. We have previously studied the convergence of the theta function. Thus *θ(x|it)* may be seen as the fundamental solution to the heat equation when the space variable x lies on a circle R/Z. Similarly, the function $\theta_1(z)$ satisfies a third-order differential equation. Indeed, it is enough to use the relation.

$$\wp(z)=-\frac{\partial^2}{\partial z^2}\log\theta_1(z)+C$$

where C is a constant ℘ (z)is the Weierstrass function defined by

$$\wp(z)=\frac{1}{z^2}+\sum_{\omega\in\wedge\setminus\{0\}}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right)$$

$\wedge=Z\omega_1\oplus Z\omega_2$ is the lattice generated by two non-zero complex numbers ω_1 and ω_2 such that: $\operatorname{Im}\left(\frac{\omega_2}{\omega_1}\right)>0$, and take into account the differential equation:

$$(\wp'(z))^2=4(\wp(z))^3-g2\wp(z)-g3, \tag{6}$$

Where

$$g_2=60\sum_{\omega\in\wedge\setminus\{0\}}\frac{1}{\omega^4},\quad g_3=140\sum_{\omega\in\wedge\setminus\{0\}}\frac{1}{\omega^6}$$

Moreover, we have the following classical identities [1-16]:

Theorem 5

The theta function satisfies the addition formulas.

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z_1+z_2)=\sum_{2\delta(z_2)^g}\hat{\theta}\begin{bmatrix}\frac{\alpha+\beta}{2}+\delta\\ \gamma+\varepsilon\end{bmatrix}(2z_1)\hat{\theta}\begin{bmatrix}\frac{\alpha-\beta}{2}+\delta\\ \gamma-\varepsilon\end{bmatrix}(2z_2)$$

where $\alpha,\beta,\gamma,\varepsilon\in R^g$

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z)\equiv\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z\mid B),\ \hat{\theta}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z)\equiv\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z\mid 2B)$$

$$\theta\begin{bmatrix}m_1\\ n_1\end{bmatrix}(z_1).\theta\begin{bmatrix}m_2\\ n_2\end{bmatrix}(z_2).\theta\begin{bmatrix}m_3\\ n_3\end{bmatrix}(z_3).\theta\begin{bmatrix}m_4\\ n_4\end{bmatrix}(z_4)$$

$$=\frac{1}{2^g}\sum_{2(a_1,a_2)\in(z_2)^{2g}}e^{-4\pi i\langle m_1,a_2\rangle}.\theta\begin{bmatrix}K_1+a_1\\ l_1+a_2\end{bmatrix}(\omega_1)\ldots\theta\begin{bmatrix}K_4+a_1\\ l_4+a_2\end{bmatrix}(\omega_4)$$

where $(z_1,\ldots,z_4)=(w_1,\ldots,w_4)M$ with

$$M=\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}$$

Here $\begin{pmatrix}m_1\\ n_1\end{pmatrix},\ldots,\begin{pmatrix}m_4\\ n_4\end{pmatrix},\begin{pmatrix}k_1\\ l_1\end{pmatrix},\ldots,\begin{pmatrix}k_4\\ l_4\end{pmatrix}$ are arbitrary vectors of order 2g with

$$\left(\begin{pmatrix}m_1\\ n_1\end{pmatrix},\ldots,\begin{pmatrix}m_4\\ n_4\end{pmatrix}\right)=\left(\begin{pmatrix}k_1\\ l_1\end{pmatrix},\ldots,\begin{pmatrix}k_4\\ l_4\end{pmatrix}\right)M,$$

and 1 denotes the unit matrix of order g or 2g.

In particular, we have the formulas:

$$\left(\theta\begin{bmatrix}0\\0\end{bmatrix}(z)\right)^2.\left(\theta\begin{bmatrix}0\\0\end{bmatrix}(0)\right)^2=\left(\theta\begin{bmatrix}0\\1/2\end{bmatrix}(z)\right)^2.\left(\theta\begin{bmatrix}0\\1/2\end{bmatrix}(0)\right)^2$$
$$+\left(\theta\begin{bmatrix}1/2\\0\end{bmatrix}(z)\right)^2.\left(\theta\begin{bmatrix}1/2\\0\end{bmatrix}(0)\right)^2,$$

And

$$\left(\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z)\right)^2.\left(\theta\begin{bmatrix}0\\0\end{bmatrix}(0)\right)^2=\left(\theta\begin{bmatrix}0\\1/2\end{bmatrix}(z)\right)^2.\left(\theta\begin{bmatrix}1/2\\0\end{bmatrix}(0)\right)^2$$
$$-\left(\theta\begin{bmatrix}1/2\\0\end{bmatrix}(z)\right)^2.\left(\theta\begin{bmatrix}0\\1/2\end{bmatrix}(0)\right)^2,z$$

as well as the identity of Jacobi obtained by posing z = 0,

$$\left(\theta\begin{bmatrix}0\\0\end{bmatrix}(0)\right)^4=\left(\theta\begin{bmatrix}0\\1/2\end{bmatrix}(0)\right)^4+\left(\theta\begin{bmatrix}1/2\\0\end{bmatrix}(0)\right)^4.$$

We shall see how to express the meromorphic functions on the torus C/Λ in terms of the theta function. Several approaches are possible:

Approach 1

Recall that any rational fraction (hence a meromorphic function on $P^1(C)$) can be written in the form

$$f(z)=\prod_{j=1}^{m}\frac{z-P_j}{z-Q_j} \tag{7}$$

By analogy, let P_1, ..., P_m,Q_1, ...,Q_m be points of the Riemann surface X and f(z) a function having zeros at the points P_1, ..., P_m and poles at points Q_1, ...,Q_m. It is assumed that condition (i) (or what is equivalent, condition (ii)) of Abel's theorem1 is satisfied. Since X has genus 1, then there exists a single holomorphic differential ω on X. Still according to Abel's theorem [1], the existence of the function f(z) imposes the condition.

$$P_1+P_2+\cdots+P_m=Q_1+Q_2+\cdots+Q_m$$

Note that for m = 1; $P_1=Q_1$ and the only valid case is f(z)=constante. In the case where m≥2, the function f(z) may be expressed in terms of *θ* as follows:

$$f(z)=C\prod_{j=1}^{m}\frac{\theta\left(z-P_j-\frac{1}{2}(1+b)\right)}{\theta\left(z-Q_j-\frac{1}{2}(1+b)\right)}, \tag{8}$$

where C is a constant. This formula may be considered as the straightforward generalization of the representation (7) for the meromorphic (rational) function on $P^1(C)$. The most important difference is that in (7) the positions of the poles and zeros are arbitrary. To verify that f(z) in (8) is indeed single valued on X we have to check that *f(z + 1) = f(z) and f(z + b) = f(z).*

The first relation is trivial and according to relation (2) and the fact that $\sum_{j=1}^{m}P_j=\sum_{j=1}^{m}Q_j$, we also have the second relation. Thus f is doubly periodic. The function *f* is meromorphic with zeros in $Q_j+\frac{1}{2}(1+b)$ and poles in $P_j+\frac{1}{2}(1+b)$.

Approach 2

The function log θ(z) can be expressed as the sum of a doubly periodic function of periods 1, b and a linear function. Therefore the function $\frac{d_2}{dz^2}\log\theta(z)$ is doubly periodic and meromorphic over X, with a double pole in $z=\frac{1}{2}(1+b)$. This function coincides with the Weierstrass function $\wp(Z)$:

$$\wp(Z)=\frac{d_2}{dz^2}\log\theta(z)+C, \tag{9}$$

Where C is a constant chosen in such a way that Laurent's series expansion of $\wp(Z)$ en Z=0 has no constant term. The connection given by (9) between the Weierstrass function $\wp(Z)$ and the theta function is obvious in view of

the right-hand side (as noted above) and the information obtained above on the location of the zeros of θ(z). Now from (9) and (6) it follows that the function (z) satises a dierential equation of 3rd order.

Approach 3

Recall that meromorphic functions with simple poles on $P^1(C)$ can be written in the form

$$f(z)=\sum_j\frac{\lambda_j}{z-P_j}+C,$$

where λ_j∈C and C is a constant. By analogy, we consider on X the function

$$f(z)=\sum_j\lambda_j\frac{d}{dz}\log\theta(z-P_j)+C,$$

where P_j∈X, λ_j∈C such that $\Sigma_j\lambda_j=0$ et C is a constant. This function is doubly periodic and meromorphic with simple poles in $P_j+\frac{1}{2}(1+b)$ and residues λ_j at these points.

We have seen how the meromorphic functions on the torus C/Λ can be expressed in terms of theta function. Moreover, for g=1, we know that: X ≃ C/Λ ≃ Jac(X). So the construction that was done previously on the torus C/Λ or what amounts to the same on Jac(X) is also valid on the Riemann surface X. For example, let us take the case of a function having poles in P_1,..., P_m and zeros in Q_1, ...,Q_m on the Riemann surface X. According to Abel's theorem, we have

$$\sum_{j=1}^{m}\varphi(P_j)=\sum_{j=1}^{m}\varphi(Q_j),$$

and it is possible, according to approach 1 described above, to express the function *f*(P) in terms of theta function using the formula

$$f(P)=C\prod_{j=1}^{m}\frac{\theta\left(\varphi(P)-\varphi(Q_j)\frac{1}{2}(1+b)\right)}{\theta\left(\varphi(P)-\varphi(P_j)\frac{1}{2}(1+b)\right)}.$$

Let us now pass to the case where the surface of Riemann X is of genus g>1. Recall that the Jacobi inversion problem [5,16] consists in determining g points P_1, ..., P_g on X such that:

$$\sum_{k=1}^{g}\int_{P_0}^{P_k}\omega\equiv z_j\quad(\mathrm{mod}\,L),\ j=1,\ldots,g$$

Where (z_1, ..., z_g) ∈ Jac(X), (ω_1,…,ω_g) is a base of holomorphic differentials on X, P_0 is a base point on X and L is a lattice generated by the column vectors of the period matrix. In other words, the problem is to determine the divisor $D\sum_{j=1}^{g}P_j$ in terms of z = (z_1, ..., z_g) 2 Jac(X) such that if ' is the Abel-Jacobi map, then the equation $\varphi(D)=Z$ is satisfied. We will study the Jacobi inversion problem using theta functions.

Theorem 6

If the function defined by

$$\zeta(P)=\theta(\varphi(P)-C),\quad C\in\mathbb{C}^g$$

is not identically zero, then it admits g zeros (counted with their order of multiplicity) on the normal representation X^* of X, denoted by the symbol $a_1b_1a_1^{-1}b_1^{-1}\ldots a_gb_ga_g^{-1}b_g^{-1}$, where ($a_1$, . . . , a_g, b_1, . . . , b_g) Is a symplectic basis of the homology group H_1(X, Z). Moreover, if P_1, ..., P_g denote the zeros of this function then we have on the Jacobian variety Jac(X) the formula.

$$\sum_{k=1}^{g}\varphi(P_k)\equiv c-\Delta,\ (\text{mod. periods})$$

Where Δ∈ $\mathbb{C}^g$ is the vector of the Riemann constants defined by

$$\Delta_j=\frac{1}{2}(1+b_{jj})-\sum_{k\neq j}\left(\int_{ak}\omega_k(P)\int_{P_0}^{P}\omega_j\right),\quad j=1,\ldots,g. \tag{10}$$

Proof:

Note that X^* is a polygon with 4g sides identified in pairs. If one traverses the boundary $\delta X^*$of this polygon, we notice that each side is traversed twice, one in the direction of its orientation and the other in the opposite direction. So δX^* may be represented as follows

$$\partial X^*=\sum_{j=1}^{g}(a_j+b_j-a_j^{-1}-b_j^{-1}).$$

To calculate the number of zeros, we have to compute the integral (logarithmic residue): $\frac{1}{2\pi i}\int_{\partial x^*}d\log\xi(P)$ We denote by ζ the value of the function ζ(p) on a_1^{-1}, b_1^{-1} and by ζ^+ the value of ζ(p) on the segments a_jb_j. We will use similar notations φ^+, φ^- for the Abel map $\varphi(p)$. In this notation the above integral can obviously be rewritten in the form

$$\frac{1}{2\pi i}\int_{\partial x^*}d\log\xi(P)=\frac{1}{2\pi i}\sum_{k=1}^{g}\left(\int_{ak}+\int_{bk}(d\log\zeta^+-d\log\zeta-\right)$$

Note that

$\varphi_j^-(\mathrm{P})=\varphi_j^-(\mathrm{P})+\mathrm{b}_{jk}$,

if P∈a_k and

$\varphi_j^+(\mathrm{P})=\varphi_j^-(\mathrm{P})+\delta_{jk}$,

if P∈b_k. From Theorem 4, we have

$d\log\varphi^-(\mathrm{P})=d\log\varphi^+(\mathrm{P})-2\pi i\omega_k$ on a_k,

$d\log\varphi^+(\mathrm{P})=d\log\varphi^-(\mathrm{P})$ sur b_k,

Therefore (2), implies

$$\frac{1}{2\pi i}\int_{\partial x}d\log\zeta=\frac{1}{2\pi i}\sum_{k=1}^{g}\int_{ak}2\pi i\omega_k=g$$

which shows that the function ζ(p) admits g zeros on X^*. To prove the second part of the theorem, we consider the integral

$$I_j=\int_{ax^*}\varphi_j(\mathrm{P})d\log\zeta(\mathrm{P}),\ j=1,\ldots,g.$$

By designating P_1, ..., P_g the zeros of the function ζ(p) and taking into account the residual theorem, we have

$$I_j = \varphi_j(P_1) + ... + \varphi_j(P_g).$$

By reasoning as before, one obtains

$$I_j = \frac{1}{2\pi i}\sum_{k=1}^{g}(\int_{ak} + \int_{bk})(\varphi_j^+ d\log\zeta^+ - \varphi_j^- d\log\zeta^-),$$

$$\frac{1}{2\pi i}\sum_{k=1}^{g}\int_{ak}(\varphi_j^+ d\log\zeta^+ - (\varphi_j^+ + b_{jk})(d\log\zeta^+ - 2\pi i\omega_K)),$$

$$+\frac{1}{2\pi i}\sum_{k=1}^{g}\int_{bk}(\varphi_j^+ d\log\zeta^+ - (\varphi_j^+ - \delta_{jk})d\log\zeta^+),$$

$$+\sum_{k=1}^{g}(\int_{ak}\varphi_j^+\omega_k - \frac{1}{2\pi i}b_{jk}\int_{ak} d\log\zeta^+ + b_{jk}) + \frac{1}{2\pi i}\int_{bk} d\log\zeta^+).$$

Note that

$$\int_{ak} d\log\zeta^+ = 2\pi \mathrm{i} n_k, n_k \in \mathbb{Z}$$

Similarly, by designating by Q_j (resp. Q_J^*) the beginning (or end) of the contour b_j, then

$$\int_{bj} d\log\zeta^+ = \log\zeta^+(Q_J^*) - \log\zeta^+(Q_j) + 2\pi \mathrm{i} m_j, m_j \in \mathbb{Z}$$

$$= \log\theta(\varphi(Q_j) + f_j - C) - \log\theta(\varphi(Q_j) - C) + 2\pi \mathrm{i} m_j,$$

$$= -\pi \mathrm{i} b + 2\pi \mathrm{i} C_j - 2\pi \mathrm{i}\varphi_j(Q_j) + 2\pi \mathrm{i} m_j,$$

where $f_j = (b_{1j} ... b_{gj})^T$, j=1, ..., g, denote the columns of the matrix B. Therefore,

$$I_j = C_j - \frac{1}{2}b_{jj} - \varphi_j(Q_j) + \sum_{k=1}^{g}\int_{ak}\varphi_j(P)\omega_k (\mathrm{mod.}\, periods).$$

The beginning of the contour aj will be designated by R_j and its end obviously, coincides with the beginning Q_j of the contour b_j. We have

$$I_j = C_j - \frac{1}{2}b_{jj} - \varphi_j(Q_j) + \int_{aj}\varphi_j(P)\omega_j + \sum_{\substack{k=1 \\ k\neq j}}^{g}\int_{ak}\varphi_j(P)\omega_k$$

$$= C_j - \frac{1}{2}b_{jj} - \varphi_j(Q_j) + \frac{1}{2}(\varphi_j^2(Q_j) - \varphi_j^2(R_j)) + \sum_{\substack{k=1 \\ k\neq j}}^{g}\int_{ak}\varphi_j(P)\omega_k,$$

$$= C_j - \frac{1}{2}b_{jj} - \varphi_j(Q_j) - 1 + \frac{1}{2}((\varphi_j^2(R_j) + 1)^2 - \varphi_j^2(R_j)) + \sum_{\substack{k=1 \\ k\neq j}}^{g}\int_{ak}\varphi_j(P)\omega_k,$$

$$= C_j - \frac{1}{2}(1 + b_{jj}) + \sum_{\substack{k=1 \\ k\neq j}}^{g}\int_{ak}\varphi_j(P)\omega_k,$$

which ends the proof.

In general, the vector Δ depends on P_0 except in the particular case g = 1, Where = $\frac{1}{2}$ (1 + b). We show that 2Δ=ϕ−(K), where K is the canonical divisor. Hence, by skillfully choosing the point P_0 we can express K in a very simple way. For example, consider the case where X is a hyperelliptic curve of genus g of affine equation.

$$\omega^2 = \prod_{j=1}^{2g+2}(\xi - \xi_j),$$

where all z_j are distinct. Let (a_1, . . . , a_g, b_1, . . . , b_g) be a symplectic basis of the homology group H_1(X, Z)) and let

σ: X → X, (ω,ξ)→(-ω,ξ),

be the hyperelliptic involution (i.e., which consists in exchanging the two sheets of the curve X) with (a_j) = $-a_j$ and σ (b_j) = $-b_j$. Note that

$$\int_{aj}\omega_k = -\int_{\sigma(aj)}\omega_k = -\int_{aj}\sigma^*\omega_k.$$

Then, by choosing $P_0 = \xi_1$, we get

$$\Delta_j = \frac{1}{2}(1 + b_{jj}) + \sum_{k\neq j}\int_{aj}\omega_k\left(\int_{\xi 1}^{\xi 2k+1}\omega_j + \int_{\xi 2k+1}^{P}\omega_j\right), j = 1, ..., g,$$

$$= \frac{1}{2}(1 + b_{jj}) + \sum_{k\neq j}\int_{\xi 1}^{\xi 2k+1}\omega_j\int_{aj}\omega_k$$

$$+\sum_{k\neq j}\int_{\xi 2k+1}^{\xi 2k+2}\left(\left(\int_{\xi 2k+1}^{p}\omega_j\right)\omega_K(p)\left(\int_{\xi 2k+1}^{\sigma p}\omega_j\right)\omega_K(\sigma p)\right)$$

Taking into account that ω_k (σP)= ω_k (P) and modulo a linear combination $n+B_m$ (a lattice generated by the column vectors of the period matrix), we obtain in this case the formula :

$$\Delta_j + \sum_{k=1}^{g}b_{jk} + \frac{j}{2}, 1 \le j \le g.$$

The zeros of a theta function on $\mathbb{C}^g$ form a submanifold of Jac(X) of dimension g − 1 called theta divisor Θ={z : θ(z) = 0}. It is invariant by a finite number of translations and can be singular. Equation (3) implies that Θis well defined on the Jacobian variety Jac(X). Since θ(z) = θ(z), we deduce that Θ is symmetric: -Θ=Θ.

Theorem 7

(Riemann). The function

ζ(P)= θ(ϕ(P) − C), C∈ $\mathbb{C}^g$,

is either identically zero, or admits exactly g zeros Q_1, ...,Q_g on X such that:

$$\sum_{k=1}^{g}\varphi(Q_j) = c + \Delta$$

where Δ is defined by (10).

This result means that when we embed the Riemann surface X into its Jacobian variety Jac(X) via the Abel map ϕ, then its image is fully include in the theta divisor, or it meets it in exactly g points. In fact, if ζ(P) is not identically zero on X, then its zeros coincide with the points P_1, ..., P_g and determine the solution of the Jacobi inversion problem ϕ(D) = z for the vector z = C −. Recall that D ∈ Div(X) is a special divisor if and only if dim L(D)≥ and dim L(K −D) ≥1where K is a canonical divisor. In the

case where D ≥ 0 a divisor is special if and only if dim Ω^1(D)≠0 Note also that the special divisors of the form

D = P_1 + · · · + P_N, N = deg D ≥ *g*,

coincide with the critical points of the Abel-Jacobi map,

$\mathrm{Sym}^N X \to \mathrm{Jac}(X),\ D \mapsto \left(\int_0^D \omega_1, ..., \int_0^D \omega_N\right),$

or what amounts to the same

ϕ (P1, ..., PN) = ϕ (P1) + · · · + ϕ (PN).

These critical points are the points P1, ..., PN where the rank of the differential of this application is less than g. From Riemann's theorem 7, the function ζ(P) = θ(ϕ (P) − C) is identically zero if and only if

C≡ϕ(Q_1) + · · · · · · + ϕ (Q_g) +Δ

where Q_1 + · · · + Q_g is a special divisor.

Theorem 8

Let $z = (z_1, ..., z_g) \in \mathbb{C}^g$ be a vector such that the function

ζ(P)= θ(ϕ(P) – *Z*–Δ),

is not identically zero on X. Then the function ζ(P) admits exactly g zeros P_1, ..., P_g on X which determine the solution of the Jacobi inversion problem

ϕ (D)=*Z*, where $D = \sum_{j=1}^{g} P_j$ i.e., the solution of

$$\varphi_j(P_1)+...+\varphi_j(P_g)=\sum_{j=1}^{g}\int_{P_0}^{P_k}\omega_j \equiv z_j, 1\leq j\leq g \tag{12}$$

(recall that the symbol≡, as usual, means congruence modulo the period lattice). Moreover, the divisor D is not special and the points P_1, ..., P_g are only determined from the system (12) up to a permutation.

Proof

The first assertion results from theorem 1. Moreover, the divisor $D=\sum_{J=1}^{g} P_j$ is not special because otherwise the function ζ(P) would be identically null from what precedes, which is absurd. For the last point, assume that the system (12) admits another solution Q_1, ...,Q_g. We will have on the Jacobian variety Jac(X),

$$\sum_{J=1}^{g}\varphi(P_j)\equiv\sum_{J=1}^{g}\varphi(Q_j),$$

where L is the lattice generated by the period matrix. According to Abel's theorem, this means that there exists a meromorphic function on X having zeros in Q_1, ...,Q_g and poles in P_1, ..., P_g. Now we have just shown that the divisor is non-special, so such a function must be a constant, which means that $P_j = Q_j$, j = 1, ..., g.

For example, if

$$D=\sum_{J=1}^{g} P_j$$

is a non-special divisor on a Riemann surface X of genus g, then the function θ(ϕ(P) – ϕ(D)–Δ), admits exactly g zeros on X at the points P = P_1, ..., P_g. We have the following characterization of the theta divisor:

Theorem 9

We have θ(C) = 0, if and only if there exists P_1, ..., P_{g-1} ∈ X with base point P0, such that:

$$C\equiv\varphi(P_1)+...+\varphi(P_{g-1})+\Delta=\sum_{J=1}^{g-1}\int_{p_0}^{p_j}\omega+\Delta.$$

Proof

Let us return to the function

ζ(P)= θ(ϕ(P) – *C*),

and assume first that it is non-zero on X. By theorem 6, this function admits g zeros P_1, ..., P_g on X and

$$C\equiv\phi(P_1)+\ldots+\phi(P_g)+\Delta. \tag{13}$$

The set of these zeros being unique and as by hypothesis θ(C) = 0, then $P_g = P_0$. Therefore ϕ(P_g)= ϕ(P_0) and from (13), we have

C≡ϕ(P_1)+…+ ϕ(P_{g-1})+Δ.

Let us now turn to the case where the function ζ(P) is not identically zero on X. According to theorem 6, we have

$$C\equiv\phi(Q_1)+\ldots+\phi(Q_g)+\Delta. \tag{14}$$

where Q_1 +· · ·+Q_g is a special divider. The latter implies the existence on X of a non-constant meromorphic function ζ having poles in Q1, ...,Qg with (P_0)=0. Then,

ϕ(P_1+…+ P_{g-1}+ P_0) ≡ϕ(Q_1+…+ Q_g),

according to the Abel theorem where P1 + · · · + Pg−1 + P0 is the divisor of the zeros of ζIt is therefore su-cient to replace in the formula (14), ϕ(Q_1+…+ Q_g), by ϕ(P_1+…+ P_{g-1}+ P_0) while taking into account that (P0) = 0.

Theorem 10

Let D be a non-special divisor of degree g, D′ a positive divisor of degree n, a positive divisor of degree n, (ω_1,…,ω_g) a base of holomorphic differentials on a base of holomorphic differentials on X, $\varphi(P)=(\int_{p_0}^{p}\omega_1,...,\int_{p_0}^{p}\omega_g)$ the Abel map with base point P_0, η normalized differential of the 3rd kind 2 on X having poles on D′ and residues −1, U = (U_1, ...,U_g) the vector-periods with $U_K = \int b_k \eta$ and finally Δ the vector defined using the Riemann constants by (10). If ψ is a meromorphic function on X having g + n poles on D + D′, then this function is expressed in terms of theta function by the formula

$$\varphi(P)=A\frac{\theta(\varphi(P)-\varphi(D)+U-\Delta)}{\theta(\varphi(P)-\varphi(D)-\Delta)}e^{\int_{p_0}^{p}\eta},\ \text{A = constante.}$$

Proof: It should be noted that the integration contour in integrals $\int_{p_0}^{p}\eta$, and $\int_{p_0}^{p}w_j, j=1,....,g$ is the same. The function ψ (P) admits poles only on D + D′. Let us show that this function is well defined on X; i.e., it does not depend on the path of integration. In other words, it does not change when P goes through any cycle $\gamma=\sum_{k=1}^{g}(n_k a_k+m_k b_k)\in H_1(X,Z)$ the expressions $\int_{p_0}^{p}\eta$, and $\varphi(P)=\left(\int_{p_0}^{p}\omega_1,.....,\int_{p_0}^{p}\omega_g\right)$ is transformed respectively as follows:

$$\int_{p_0}^{p}\eta+\sum_{k=1}^{g}m_k\int_{b_k}\eta=\int_{p_0}^{p}\eta+2i\langle m,U\rangle,\quad m=(m_1,....m_g)\in\mathbb{Z}^g,$$

and Moreover, using formula (4), one obtains

$$\frac{\theta(\varphi(P)-\varphi(D)+U-\Delta}{\theta(\varphi(P)-\varphi(D)-\Delta}=\frac{e^{-\pi i\langle Bm,m\rangle-2\pi i\langle m,\varphi(P)-\varphi(D)+U-\Delta\rangle}}{e^{-\pi i\langle Bm,m\rangle-2\pi i\langle m,\varphi(P)-\varphi(D)-\Delta\rangle}}=e^{-2\pi i\langle m,U\rangle},$$

and the result follows from the above transformation.

On the Riemann surface X of genus g, singular functions possessing g poles and essential singularities play a crucial role in the study of integrable systems, in particular the Korteweg-de Vries equation K-dV),

$$\frac{\partial u}{\partial t}-6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0,$$

the Kadomtsev-Petviashvili equation (KP),

$$\frac{\partial^2 u}{\partial y^2}-\frac{\partial}{\partial x}\left(4\frac{\partial u}{\partial t}-12u\frac{\partial u}{\partial x}-\frac{\partial^3 x}{\partial x^3}\right)=0,$$

the nonlinear Schrödinger equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u^2}{\partial x^2}=0,$$

the Boussinesq equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u^2}{\partial x^2}=0,$$

the Camassa-Holm equation

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial t\partial x^2}+3u\frac{\partial u}{\partial x}=2\frac{\partial u\partial^2 u}{\partial x\partial x^2}+u\frac{\partial^3 u}{\partial x^3},$$

whose exact solutions are solitons [14], i.e., self-reinforcing solitary waves that maintain their shape as they propagate at a constant rate. We shall see by analogy from the previous theorem how to express these functions (known as Baker-Akhiezer functions) in terms of theta functions and at the same time prove their existence. Let Q_1, ...,Q_n be points on a Riemann surface X of genus g and z_j are local parameters such that : $z_j(Q_j)=\infty$. We associate to each point Q_j an arbitrary polynomial denoted by $q_j(z_j)$.

Let

$D = P_1 + \cdots + P_g$,

be a positive divisor on X and ψ(P) a function (called Baker-Akhiezer function) satisfying the following conditions :

(i) ψ(P) is meromorphic on X\ {Q_1, ...,Q_n} and admits poles only at the points P_1, ..., P_n of the divisor D.

(ii) the function ψ(P) $e^{-q_j(Z_j(P))}$ is analytic in the neighbourhood of Q_j ,j=1, ..., n.

The condition (ii) can be replaced by this condition: the function ψ admits an essential singularity of the form ψ(P) $\sim ce^{-q_j(Z_j(P))}$ at the points Q_j, j = 1, ..., n where c is a constant. These functions ψ(P) form a vector space that we note L≡ L(D;Q_1, ...,Q_n, q_1, ..., q_n).

Theorem 11

Let $D = P_1 + \cdots + P_g$ be a non-special divisor of degree g. Then the space L is of dimension 1 and its basis is described using

$$\varphi_1(P)=\frac{\theta(\varphi(P)-\varphi(D)+V-\Delta)}{\theta(\varphi(P)-\varphi(D)-\Delta}e^{\int_{p_0}^{p}\eta}, \quad (15)$$

Where ηis a normalized differential of the 2nd kind3 having poles at the points Q_1, ...,Q_n, the main parts coincide with the polynomials $q_j(z_j)$, where j = 1, ..., n, V = (V_1, ..., V_g) with $V_k = \int b_k$ η, k = 1, ..., g,

$$\varphi(P)=\left(\int_{p_0}^{p}\omega_1,\ldots,\int_{p_0}^{p}\omega_g\right),$$

the Abel map with base point P_0, Δ is the vector defined using the Riemann constants by (10). The integration contour in integrals $\int_{p_0}^{p}\eta$, and $\int_{p_0}^{p}\omega_j$, j = 1, ..., g is the same.

Proof: The function ψ_1(P) has poles on the divisor D and essential singularities at the points Q_1, ...,Q_n. The function ψ_1(P) is well defined; It does not depend on the integration path. Using the notations and reasoning similar to those of theorem 10, we obtain

$$\frac{\theta(\varphi(P)-\varphi(D)+V-\Delta}{\theta(\varphi(P)-\varphi(D)-\Delta}=e^{-2\pi i\langle m,v\rangle},$$

and the result follows from the transformation used in the proof of the preceding theorem. Moreover, according to the Riemann-Rock theorem [5,15], the dimension of the space L is equal to deg $D-g+1$. As deg D=g, then the dimension of the space in question is equal to 1, which proves the uniqueness of the function with ψ_1 (to a multiplicative constant). Let $\psi_1\in$L be any function. Therefore, the quotient $\frac{\psi}{\psi_1}$ is a meromorphic function with g(= degD) poles. The divider of the poles of $\frac{\psi}{\psi_1}$ coincides with the divisor $D'=P_1'+\ldots+P_g'$ of the zeros of ψ_1(P)

and we must have ϕ(D′)-ϕ(D)=Vy choosing the polynomials q_j with sufficiently small coefficients (or what amounts to the same, the vectors of V sufficiently small), then the theta function which is in the numerator of the above expression is not identically zero. Therefore, its divisor D′ of the poles is not special and therefore $\frac{\psi}{\psi_1}$ is a constant.

Examples

It is well known that the solutions of many integrable systems are given in terms of theta functions associated with compact Riemann surfaces. We will see below some solutions for some interesting problems.

As a first example, we consider the motion of a solid in a perfect fluid described using the Kirchhoff equations [3]:

$$\dot{P}=p\Lambda\frac{\partial H}{\partial l},$$

$$i=p\Lambda\frac{\partial H}{\partial p}+l\Lambda\frac{\partial H}{\partial l},$$

Where, p=(p_1, p_2, p_3) ∈ R^3, l = (l_1, l_2, l_3) ∈ R^3 and H is the Hamiltonian. This system has the following three first integrals:

H_1=H,

$$H_2=P_1^2+P_2^2+P_3^2,$$

$$H_3=p_1l_1+p_2l_2+p_3l_3.$$

Two cases can be distinguished: case of Clebsch and case of Lyapunov- Steklov. In the case of Clebsch,

$$H=\frac{1}{2}\sum_{k=1}^{3}(a_kp_k^2+b_kl_k^2),$$

with condition

$$(a_2-a_3)b_1^{-1}+(a_3-a_1)b_2^{-1}+(a_1-a_2)b_3^{-1}=0.$$

The above system is written in the form of a Hamiltonian vector field. A fourth first integral is given by

$$H_4=\frac{1}{2}\sum_{k=1}^{3}(b_kp_k^2+\ell l_k^2),$$

where is a constant such that

$$\rho=b_1(b_2-b_3)(a_2-a_3)^{-1}=b_2(b_3-b_1)(a_3-a_1)^{-1}=b_3(b_1-b_2)(a_1-a_2)^{-1}$$

The method of resolution obtained by Kötter [9] is extremely complicated and relies on an astute choice of two variables s_1 and s_2. Using the substitution $b_k\to\rho b_k, 1\le k\le 3$ and an appropriate linear combination of H_1 and H_2, we can rewrite the above equations in the form

$$p_1^2+p_2^2+p_3^2=A$$

$$b_1p_1^2+b_2p_2^2+b_3p_3^2+l_1^2+l_2^2+l_3^2=B$$

$$b_1l_1^2+b_2l_2^2+b_3l_3^2-b_2b_3p_1^2-b_1b_3p_2^2-b_1b_2p_3^2=c$$

$$p_1l_1+p_2l_2+p_3l_3=D$$

where A, B, C et D are constants. Let us introduce coordinates ϕ_k, ψ_k, 1≤k≤3 by setting

$$\psi_k = p_k T_{+1} + l_k S_{+1},$$

and

$$\psi_k = p_k T_{-1} + l_k S_{-1},$$

where

$$T_{\pm 1} = \frac{\sqrt{\prod_{j=1}^{3}(z_1 - b_j)}}{\sqrt{z_1 - b_k}\sqrt{\frac{\partial R}{\partial_{z1}}}} + \frac{\sqrt{\prod_{j=1}^{3}(z_2 - b_j)}}{\sqrt{z_2 - b_k}\sqrt{\frac{\partial R}{\partial_{z2}}}}$$

$$S_{\pm 1} = \frac{\sqrt{z_1 - b_k}}{\sqrt{\frac{\partial R}{\partial_{z1}}}} + i\frac{\sqrt{z_1 - b_k}}{\sqrt{\frac{\partial R}{\partial_{z2}}}}, \quad R(z) = \prod_{i=1}^{4}(z - z_i)$$

and z_1, z_2, z_3, z_4 are the roots of equation

$$A^2\left(z^2 - z\sum_{k=1}^{3} b_k\right) + Bz - C + 2D\sqrt{\prod_{k=1}^{3}(z - bk)} = 0.$$

Let s_1 and s_2 be the roots of equation

$$\psi_1^2(v_1^2 - s)^{-1} + \psi_2^2(v_2^2 - s)^{-1} + \psi_3^2(v_3^2 - s)^{-1} = 0,$$

Where

$$v_k = \left(\frac{\sqrt{z_3 - b_k}}{\sqrt{\frac{\partial R}{\partial_{z3}}}} + \frac{\sqrt{z_4 - b_k}}{\sqrt{\frac{\partial R}{\partial_{z4}}}}\right)\left(\frac{\sqrt{z_1 - b_k}}{\sqrt{\frac{\partial R}{\partial_{z1}}}} + \frac{\sqrt{z_2 - b_k}}{\sqrt{\frac{\partial R}{\partial_{z2}}}}\right)^{-1}, \quad 1 \le k \le 3.$$

We can express the variables $p_1, p_2, p_3, l_1, l_2, l_3$ in terms of s1 and s2 [9]. After some algebraic manipulations, we obtain

$$\dot{s}_1 = \frac{(as_1 + b)\sqrt{P_5(s_1)}}{s_2 - s_1},$$

$$\dot{s}_2 = \frac{(as_2 + b)\sqrt{P_5(s_2)}}{s_1 - s_2},$$

where a, b are constants and $P_5(s)$ is a polynomial of degree five having the following form:

$$P_5(s) = s(s - v_1^2)(s - v_2^2)(s - v_3^2)(s - v_1^2 v_2^2 v_3^2).$$

Consequently, the integration is done by means of hyperelliptic functions of genus 2 and the solutions can be expressed in terms of theta functions. The problem of this motion is a limit case of the geodesic flow on SO(4). Let us remind that for an algebraically completely integrable system [1], we require that the invariants of the deferential system be polynomial (in appropriate coordinates) and that the complex manifolds obtained by equating these polynomial invariants with generic constants form the afine part of a complex algebraic torus (Abelian variety) in such a way that the complex flow generated by the invariants are linear on these complex tori. Meromorphic solutions dependent on a sufficient number of free parameters play a crucial role in the study of these systems. We show [1,6] that the differential system.

in question is algebraically completely integrable and the corresponding flow evolves on an Abelian surface

where the lattice is generated by the period matrix.

$$\Omega = \begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}, \mathrm{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0, (a,b,c \in \mathbb{C})$$

The affine surface M_c defined by putting the invariants of the system equal to generic constants, can be completed into a non-singular compact complex algebraic variety (Abelian surface) $\tilde{M}_c = M_c \cup D$ by adjoining at in infinity a smooth curve D of genus 9. The latter is a double cover of an elliptic curve ε ramified over 16 points. The application

$$\tilde{M}_c \to \mathbb{C}P^7, (t_1, t_2) \mapsto [1, X_1(t_1,t_2), ..., X_7(t_1,t_2)],$$

is an embedding of $\tilde{M}_c$ into $\mathbb{C}P^7$ where $(1, X_1, ..., X_7)$ forms a base of the space L(D) of meromorphic functions with at worst a simple pole along D (The functions $X_1, ..., X_7$ are expressed in a simple way as a function of $x_1, ..., x_6$). The solutions of the differential system in question in terms of theta functions are given by

$$X_k(t) = \frac{\theta_k\left[\left(t_1^0, t_2^0\right) + t(n_1, n_2)\right]}{\theta_0\left[\left(t_1^0, t_2^0\right) + t(n_1, n_2)\right]}, \quad k = 1, ..., 7$$

where $(\theta_0, ..., \theta_7)$ forms a base of the space of theta functions associated with D. The two functions theta $\theta_{0, 7}$ are odd while the six functions theta $\theta_1, ..., \theta_6$ are even. In the case of Lyapunov-Steklov,

$$H_1 = H = \frac{1}{2}\sum_{k=1}^{3}(a_k p_k^2 + b_k l_k^2) + \sum_{k=1}^{3} c_k p_k l_k,$$

$a_1 = A^2 b_1 (b_2 - b_3)^2 + B$, $a_2 = A^2 b_2 (b_3 - b_1)^2 + B$, $a_3 = A^2 b_3 (b_1 - b_2)^2 + B$,

$c_1 = Ab_2 b_3 + C$, $c_2 = Ab_1 b_3 + C$, $c_3 = Ab_1 b_2 + C$, where A, B and C are constants. A fourth first integral is given by

$$H_4 = H = \frac{1}{2}\sum_{k=1}^{3}(d_k p_k^2 + l_k^2) - A\sum_{k=1}^{3} d_k p_k l_k),$$

where $d_1 = A^2 (b_2 - b_3)^2$, $d_2 = A^2 (b_3 - b_1)^2$, $d_3 = A^2 (b_1 - b_2)^2$. A long and delicate calculation [10] shows that in this case too, the integration is performed using hyperelliptic functions of genus two and the solutions can be expressed in terms of theta functions. Another interesting example concerns the Landau-Lifshitz equation [2,11]:

$$\frac{\partial S}{\partial t} = S \times \frac{\partial^2 S}{\partial x^2} + S \times JS,$$

where $S = (S_1, S_2, S_3)$, $s_1^2 + s_2^2 + s_3^2 = 1$ and $J = \mathrm{diag}(J_1, J_2, J_3)$. This equation describes the effects of a magnetic field on ferromagnetic materials. The real solutions (with magnetic anisotropy of the axis of easy magnetization type) are given by

$$S_1 = \frac{\theta(\omega + d + m)\theta(\omega + d + m + r) - \theta(\omega + d)\theta(\omega + d + r)}{\theta(\omega + d)\theta(\omega + d + m + r) - \theta(\omega + d + r)\theta(\omega + d + m)}$$

$$S_2 = -i\frac{\theta(\omega + d + m)\theta(\omega + d + m + r) - \theta(\omega + d)\theta(\omega + d + r)}{\theta(\omega + d)\theta(\omega + d + m + r) - \theta(\omega + d + r)\theta(\omega + d + m)},$$

$$S_3 = -\frac{\theta(\omega + d)\theta(\omega + d + m + r) + \theta(\omega + d + r)\theta(\omega + d + m)}{\theta(\omega + d)\theta(\omega + d + m + r) - \theta(\omega + d + r)\theta(\omega + d + m)}.$$

Here the theta function is linked to a hyperelliptic curve of genus g, $\omega^2 = (\lambda^2 - a^2)\prod_{j=1}^{2g}(\lambda - e_j)$ whose cycle $\sum a = a1 + ... + a_g$ encircles the cut [1a,a], $r = \int_{\infty^+}^{\infty^-} du$ where the integration path crosses the cycle $\sum a$, the vector $d \in \mathbb{C}^g$ is such that : Im $d = -\frac{1}{2}\mathrm{Im} r$, $m = (m1, ...mg)$, $\omega = \frac{1}{2\pi}(Vx + Wt)$.

We cite yet another example which concerns the sine-Gordon equation [2]:

$$\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial^2\varphi}{\partial t^2}=\sin\varphi.$$

It is a non-linear wave equation with multiple applications in physics. Its solution can be written in the form

$$\varphi(x,t)=2i\ln\frac{\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(Ux+Vt+W\setminus B)}{\theta\begin{bmatrix}0\\ 0\end{bmatrix}(Ux+Vt+W\setminus B)}+C+2\pi m,$$

où U, V,W ∈ $\mathbb{C}^g$, C∈ R,m∈ Z.

Moreover, the study of the theta functions of a Riemann surface of the genus g can be done from the point of view of tau function of a hierarchy of soliton equations [13]. The tau functions are specific functions of time, constructed from sections of a determinant bundle on a Grassmannian manifold of in definite dimension and generalize the Riemann theta functions.

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