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On Statistically (I) - sequential spaces

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Abstract

In this paper, we answer the question "Is the product of two statistically sequential spaces statistically sequential?" raised by Zhongbao and Fucai. By an example we see that products of two statistically sequential spaces need not be statistically sequential. Also, we give a necessary and sufficient condition for the product to be in most general case of I-sequential spaces. And we point out the error in proposition 2.2 of. Further, we prove that a subspace of a statistically sequential space need not be statistically sequential and we find a necessary and sufficient condition for a subspace to be in most general case of an I-sequential space. Finally, we develop the properties of the most general concept of I-sequential spaces.

Keywords

Phrases statistical convergence; I-convergent sequence; Sequential space

Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [1,2] and Schoenberg [3]. If K⊂ℕ, then K_n will denote the set {K: k≤n} and |Kn| stands for the cardinality of *K*: The natural density of K is defined by

$$d(K)=\lim_{n\to\infty}\frac{|K_n|}{n},$$

if the limit exists [4,5]. A sequence $\{X_n\}$ in a topological space X is said to converge statistically [6] (or shortly s-converge) to $x\in X$, if for every neighborhood U of *x*, $d(\{n\in N: x_n\in U\})=1$, Any convergent sequence is statistically convergent but the converse is not true [7]. But in general, s-convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces. It has been discussed and developed by many authors [8-16].

The concept of I-convergence of real sequences [17,18] is a generalization of statistical convergence which is based on the structure of the ideal I of subsets of the set of natural numbers. In the recent literature, several works on I-convergence including remarkable contributions by salat et al. have occurred [15-20]. The idea of I-Convergence has been extended from the real number space to a topological space [14] and to a normed linear space [21].

I-convergence coincides with the ordinary convergence if I is the ideal of all finite subsets of ℕ and with the statistical convergence if I is the ideal of subsets of ℕof natural density zero.

Throughout this paper, (X, τ) will stand for a topological space and I for a nontrivial ideal of ℕ, the set of all positive integers. $X_n\to x$ denotes a sequence$\{X_n\}$ converging to *x* Let *X* be a space and P⊂X A sequence $\{X_n\}$ converging to *x* in *X* is eventually in *P* if $\{X_n|n>k\}\cup\{x\}\subset P$ for some $K\in\mathbb{N}$: it is frequently in P if $\{x_{n_k}\}$ is eventually in *P* for some subsequence $\{x_{n_k}\}$ of$\{X_n\}$. Let P be a family of subsets of *X*. Then ∪**P** and ∩**P** denote the union $\cup\{P|P\in\mathbf{P}\}$, and the intersection $\cap\{P|P\mathbf{P}\}$, respectively. We recall the following definition [22].

Definition 1

If *X* is a nonvoid set, then a family of sets is $I\subset 2^X$ is an ideal if (i) A, B∈I implies A∪B∈I and (ii)A ∈I, B⊂A, implies B∈I.

The ideal is called nontrivial if $I\neq\varnothing$ and $X\notin I$. A nontrivial ideal I is called admissible if it contains all the singleton sets. Several examples of nontrivial admissible ideals may be seen in [17].

Definition 2

A sequence $\{X_n\}$. in X is said to be I-convergent to $x_0\in X$ if for any nonvoid open set *U* containing x_0, $\{n\in\mathbb{N}: x_n\notin U\}\in I$ We call x_0 as the I-limit of the sequence $\{X_n\}$[14] .

Definition 3

Let *X* be a space. $P\subset X$ is called a sequential neighborhood of *x* in *X*, if each sequence convergent to *x*∈X is eventually in *P. A* subset *U* of *X* is called sequentially open if *U* is a sequential neighborhood of each of its points. *X* is called a sequential space [24] if each sequentially open subset of *X* is open.

Definition 4

O is I-sequentially open if and only if no sequence in *X* \O has an I-limit in O [1].

Definition 5

A subset *A* of a space *X* is said to be I-sequentially closed set if for every sequence $\{X_n\}$in *A* with $\{X_n\}$I-converges to *x*, then *x*∈A [1].

Definition 6

A topological space is I-sequential when any set O is open if and only if it is I-sequentially open.

Definition 7

A space *X* is called locally I-sequential space if every point of *x*∈X has a neighborhood which is an I-sequential space.

Even though we mainly deal with I-sequential spaces, we see the basic definition for s-sequential space since it will be useful for the examples which deal with s-sequential spaces. An I-sequential space *X* is statistically sequential if $I=\{A\subset X: d(A)=0\}$.

Definition 8

A space *X* is called statistically sequential(or shortly s-sequential)

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space [6] if for each non-closed subset A⊂X, there is a point $x\in X\setminus A$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in Astatistically converging to x.

There is another way to define s-sequential space.

Definition 9

A subset A of a space X is said to be a statistically sequentially open set (s-sequentially open) [25] if for any sequence $\{X_n\}$ statistically converge to x and $x\in A$ then $|\{n: x_n \in A\}|=\omega$

A topological space is s-sequential when any set O is open if and only if it is s-sequentially open.

Definition 10

A subset K of the set $\mathbb{N}$ is called statistically dense [6] if $d(k)=1$.

Definition 11

A subsequence S of the sequence L is called statistically dense in L [4] if the set of all indices of elements from S is statistically dense.

Remark 12

1. The limit of an I-convergent sequence is uniquely determined in Hausdorff spaces [14,6] .

2. If a sequence $(x_n)_{n\in N}$ converges to x in the usual sense, then it statistically converges to x . But the converse is not true in general.

3. A sequence$(x_n)_{n\in N}$ is statistically convergent if and only if each of its statistically dense subsequence is statistically convergent.

4. If a sequence$\{X_n\}$ I-converges to x , then every subsequence $\{x_{n_k}\}_{n_k\in\mathbb{N}\setminus I'}$ is I-convergent for every $I'\in I$

Lemma 13

Let X be a topological space and $A\subset X$ Then the following hold [1].

(a) A is I-sequentially open.

(b) $X\setminus A$ is I-sequentially closed.

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In this section, we answer the question 2 [25]: Is the product of two s-sequential spaces s-sequential?

By the Example 1 that product of two s-sequential spaces need not be s-sequential. Also, we give the necessary and sufficient condition for the product to be in most general case of I-sequential space. And we point out the error in proposition 2 of [1].

Proposition 1

The disjoint topological sum of any family of I-sequential spaces is I-sequential.

Proof. Let X be the disjoint sum of the family $\{X_i\}_{i\in\Lambda}$ of I-sequential spaces.

If U is not open in X, then for some $i\in\Lambda$ $U\cap X_i$ is not open. Since X_i is an I-sequential space, $U\cap X_i$ is not I-sequentially open. Thus, there is a point $X_i\in U\cap X_i$ and a sequence $\{x_{i_n}\}\subset X_i\setminus U$ I-converges to X_i and therefore in X. Hence U is not I-sequentially open. Therefore, X is an I-sequential space.

The following Example 1 shows that the products of two s-sequential spaces need not be s-sequential.

Example 1

For each $n\in\mathbb{N}$ let $S_n=\{x_{n,m}:m\in\mathbb{N}\}\bigcup\{x_n\}$ be a sequence statistically converges to x such that $x_{n,m}\neq x_{n,l}$ if $m\neq l$. Take $\infty\notin\bigcup\{S_n:n\in\mathbb{N}\}$. Let X' be the disjoint topological sum of S_n where $n\in\mathbb{N}$ and X be the space obtained from X' by identifying x_n to ∞. Now let f: $X'\to X$ be a natural mapping. Since each S_n is a s-sequential space and hence X' by Proposition 1. By Theorem 1 in [25], X is a s-sequential space.

$X=\bigcup\{S_n\setminus\{x_n\}:n\in\mathbb{N}\}\bigcup\{\infty\}$.

The open set in X is as follows :

1. Each point $x_{n,m}$ is isolated;

2. Each open neighborhood of the point ∞ is a set V of the form

$V=\bigcup\{M_n:n\in\mathbb{N}\}\bigcup\{\infty\}$,

where each M_n is a dense subsequence of S_n. Next we define Y: Let $Y_i=\bigcup_{n=i}^{\infty}\{S_n\}$. Now let $Y=\bigcup_{n=1}^{\infty}\{Y_n\times\{n\}\}\bigcup\{y\}$.

Topologize Y as follows:

Let each point of $\bigcup_{n=1}^{\infty}\{Y_n\times\{n\}\}$ be open and $\{V_n(y)\}$ be a countably local base at y, where $V_n(y)=\bigcup_{i\geq n}^{\infty}\{Y_i\times\{i\}\}\bigcup\{y\}$. Then Y is a metric space.

Now let $A=\{(x,(x,n))\in X\times Y/n\in\mathbb{N},x\in X_n\}$. Then $(\infty,y)\in\overline{A}\setminus A$. Thus, A is not closed in X, Y, but statistically sequentially closed in X x Y since A has no non trivial s-convergent sequence. Suppose A has a s-convergent sequence $(x_{n,}(x_{n,}i))$ statistically converges to $(,y)$, Then $\pi_1(x_{n,}(x_{n,}i))$ statistically converges to ∞ which implies for some m, $d(\{n\in\mathbb{N}/x_n\in S_m\})=k>0$. But the corresponding sequence in X, $\pi_2(x_{n,}(x_{n,}i))$ statistically converges to y which is a contradiction to $d(\{n\in\mathbb{N}/(x_n i)\in V_{m+1}(y)\})<1-k<1$. Therefore, X Y is not a s-sequential space.

Next we see the necessary and sufficient condition for the product of I-sequential spaces to be I-sequential.

Proposition 2

Every I-sequential space is a quotient of a topological sum of I-convergent sequences.

Proof. Let X be an I-sequential space. For each $x\in X$ and for each sequence $\{S_n\}$ in XI-converging to x, let $I(S,x)=\{s_n/n=1,2,3,\ldots\}\{x\}$ be a topological space, where each s_n is a discrete point and neighborhood U of x is such that $\{n\in\mathbb{N}:s_n\neq U\}\in I$. Let $X^*=\oplus_{s\in\Im}I(S,x)\times\{S\}$ where $\Im$ be the set of all I-convergent sequences. Now we consider a mapping f:$X^*\to X$ by $f((x_m,S))=x_m$.

1. f is onto.

For each point $x\in X$ there is a constant sequence S in X such that $I(S,x)=\{s_n/n=1,2,3,\ldots\}\cup\{x\}$ that is, there exists $I(S,x)X\{s\}\subset X^*$ and $f((x,S))=x$ Therefore, f is onto.

2. f is continuous.

Let U be an open set in X and $(x',S)\in f^{1}(U)$. Then there is a sequence S in X such that $x'\in I(S,x)=\{s_n/n=1,2,3,\ldots\}\cup\{x\}$ that is, $(x',S)\in I(S,x)x\{s\}\subset X^*$ and $f((x',S))=x'$. If (x',S). is an isolated point, then there is nothing to prove. If $(x',S)=(x,S)$,then there exists $I'\in I$

such that $S_n \in U$ for $n \in \mathbb{N} \setminus I'$ and hence $\{(s_n, S)/n \in \mathbb{N}/I'\} \subset f^{-1}(U)$ which is open in $I(S, X)$ and hence open in X^*. Therefore, $f^{-1}(U)$ is open in X^*. Therefore, f is continuous.

3. f is quotient.

Suppose $U \subset X$ and $f^{-1}(U)$ is open in X^*. If $x_0 \in U$ and $\{S_n\}$ is I-convergent to x_0 in X, then $(x_0,S) \in f^{-1}(U) \cap (I(S,x_0) x\{S_n\})$ which is an open neighborhood of (x_0,S) in $I(S,x_0)x\{S\}$). Then as a subset of $I(S,x_0)$ x$\{S\}$, there exists $I' \in I$ such that $(S_n,S) \in f^{-1}(U) \cap (I(S,x_0)x\{S_n\}) \subset f^{-1}(U)$ for $n \in \mathbb{N} \setminus I'$ and so $S_n \in U$ for $n \in \mathbb{N} \setminus I'$. Hence U is I-sequentially open and thus open. Therefore, f is a quotient mapping.

Here $I(S_n,x)$ is not second countable, when the set of all finite subsets of ℕ is a proper subset to I and not a Hausdorff space, when the set of all finite subsets of ℕ properly contains I. Hence for such I, $I(S_n,x)$ is not a metrizable space and hence its topological sum.

In Proposition 2 in [1], $S = \oplus\{x_n\}_{n\in\mathbb{N}} \times Y$ is mentioned as a metric space. But it is true only when I is the set of all finite subset of ℕ.

Also, the following Proposition 3 is not true in general.

Proposition 3

Every I-sequential space X is a quotient of some metric space.

Suppose Proposition 3 is true. Then by Proposition 2 [1], X is an I-sequential space if it is a quotient image of a metric space. And hence I-sequential space and sequential space coincide, by Corollary 1.14 in [23] which is a contradiction to Example 2 in [25].

Lemma 2.5

Let X be an I-sequential space. If $f: X \to Y$ preserves I-convergence sequence, then f is continuous.

Proof. Let U be an open subset in Y. Since X is an I-sequential space, it is enough if we prove that $f^{-1}(U)$ is an I-sequentially open subset of X.

Let $\{X_n\}$be a sequence in X which I-converges to $x \in f^{-1}(U)$ By our assumption, $\{f(x_n)\}$I-converges to $f(x) \in U$ Since U is open in Y, for some $I' \in I$, $f(x_n) \in U$ for $n \in \mathbb{N} \setminus I'$. This implies $x_n \in f^{-1}(U)$ for $n \in \mathbb{N} \setminus I'$. and hence $f^{-1}(U)$ is an I-sequentially open subset of X. Therefore, f is continuous.

Theorem 1

Let X^* and Y^* be the spaces obtained from X, Y as in Proposition 2. Then the following hold

1. X^* is an I-sequential space
2. $(XxY)^*$ is homeomorphic to X^*xY^*
3. X^*xY^* is an I-sequential space.

Proof 1

Let U be an I-sequential open subset of X^* and $(x',S) \in U$ where S is an I-convergent sequence $\{x_n\}$in X with its limit x and $x \in S$ This implies (x_n, S) I-converges to (x, S) in X^*. If $(x',S) = (x_n,S)$ for some n, then there is nothing to prove. If $(x',S) = (x_n,S)$ then $\{n: (x_n,S) \notin U\} \in I$ since U is an I-sequentially open subset of X^*. This implies $\{(x_n,S)/(x_n,S) \in U)\}$ is open in X^* which is contained in U. Therefore, U is open in X^*. Therefore, X^* is an I-sequential space.

Proof 2

Let f: $(XxY)^* \to X^*xY^*$ be defined by $f((x,y),s)) = ((x,\pi_1(s)), (y,\pi_2(s))$ where S is an I-convergent sequence with its limit in XxY and $(x,y) \in S$. Since projection mappings π_1,π_2 are continuous and by Proposition 1 in [1], $\pi_1(s)$ and $\pi_2(s)$ are I-convergent sequences in X and Y. respectively. Therefore, f is well defined.

(a) f is one-one and onto

Since product of I-convergent sequence is again I-convergent sequence, f is onto. Also, we easily check that f is one-one.

(b) f is continuous

Clearly, f preserve I-convergence and by Theorem, f is continuous.

(c) f^{-1} is continuous

Let U be an open set in $(XxY)^*$ and $((y',S'),(y'',S'')) \in (f^{-1})^{-1}(U)$. If y' and y'' are the limit point of S' and S'', respectively, then $S' \times S''$ I-converges to (y', y''). This implies that there exists $i \in$I such that $((y_n', y_n''), S) \in U$ for all $n \in \mathbb{N} \setminus I$. Now let $V = \{\{y_n' / n \in \mathbb{N} \setminus I\} \cup \{y'\}\} \times \{S'\}$ is open in X^* and $W = \{\{y_n'' / n \in \mathbb{N} \setminus I\} \cup \{y''\}\} \times \{S''\}$ is open in Y^*. This implies $((y',S'),(y'',S'')) \in V \times W \subset (f^{-1})^{-1}(U)$. For y', y'' are not a limit point, we can easily find an open set in $(f^{-1})^{-1}(U)$. Therefore, $(f^{-1})^{-1}(U)$ is open in X^*xY^*.

Therefore, $(XxY)^*$is homeomorphic to X^*xY^*.

3. By (a), $(XxY)^*$is an I-sequential space. By(b) and Proposition 2.1 in [1], X^*xY^* is an I-sequential space.

By Lemma 3 and Definition 6, we have the following Theorem 2.

Theorem 2

Let X be a topological space. Then the following are equivalent:

(a) X is an I-sequential space.

(b) Every I-sequentially open subset of X is open.

(c) Every I-sequentially closed subset of X is closed.

Theorem 3

Each I-sequentially open (closed) subspace of an I-sequential space is I-sequential.

Proof. Let X be an I-sequential space. Suppose that Y is an I-sequentially open subset of X. By Theorem 2, Y is open in X. Let U be an arbitrary I-sequentially open subset in Y and let $\{x_n\}$ be a sequence in X I-converges to $x \in U$ Then $x \in Y$ and since Y is an I-sequentially open subset of X. there exists $I' \in I$ such that

$\{x_n : n \in \mathbb{N} \setminus I'\} \subset Y$. Then by Remark 1.12, $\{x_{n_k}\}_{n_k \in \mathbb{N} \setminus I'}$ is an I-convergent subsequence in Y. Since U is an I-sequentially open subset in Y, there exists $I'' \in I$ such that $\{x_{n_k}\}_{n_k \in \mathbb{N} \setminus I''}$ is in U. That is, $\{x_n\}_{n \in \mathbb{N} \setminus (I' \cup I'')}$ is in U. Therefore, U is I-sequentially open in X. and hence open in X. This implies U. is open in Y, since Y is open in X. Therefore, Y is an I-sequential space.

If Y is an I-sequentially closed subset of X, then by Theorem 2, Y is closed in X. Let F be an I-sequentially closed subset in Y and let $\{x_n\}$ be a sequence in F I-convergent to x Since Y is closed, $x \in Y$ and hence $x \in F$. Therefore, F is an I-sequentially closed set in X and hence it is closed in X. Since Y is closed in X, F is closed in Y.

Proposition 4

Every locally I-sequential space X is I-sequential.

Proof. Let U be an I-sequentially open set in X. Since X is a locally I-sequential space, there exists an I-sequential neighborhood V of x With out loss of generality, assume that V is open and an I-sequential space. Since U is I-sequentially open, $U \cap V$ is an I-sequentially open subset of V. Since U is an I-sequential space, $x \in U \cap V$ is open in V and hence in X which is contained in U. Therefore, U is open and hence X is an I-sequential space.

In the rest of the section, the space X^* and the mapping $\phi = f: X^* X$ are as in Proposition 2.3.

Theorem 4

The product of two I-sequential spaces X and Y is I-sequential iff $\phi_X \, x \phi_Y$ is a quotient mapping.

Proof. Suppose $\phi_X \, x \phi_Y$ is a quotient mapping. By (c) in Theorem 1, $X^* x Y^*$ is I-sequential. By Proposition 2 in [1] and by assumption, XxY is I-sequential.

Conversely, suppose that XxY is I-sequential. Then $\phi_{XxY}(XxY)^* \to XxY$ is a quotient mapping (in proof of Proposition 3) and f^{-1}: $X^* x Y^* \to (XxY)^*$ is also a quotient mapping (defined in proof of Theorem 1 (b)). Since composition of two quotient mapping is again quotient, $\varphi_{X\times Y} \circ f^{-1} = \varphi_X \times \varphi_Y : X^* \times Y^* \to X \times Y$ is a quotient mapping.

Since a s-sequential space is not closed under Cartesian product like a sequential space, naturally, one can arise a question that " Is subspace of a s-sequential space, s-sequential?". The answer is not as shown by the following Example 2.

Example 2

Let S_n be the s-convergent sequence with its limit a_n for eachn $n \in \mathbb{N}$ that is $S_n = \{x_{n,m} : m \in \mathbb{N}\} \bigcup \{a_n\}$ and $\{x_n\}$be a s-convergent sequence with its limit x. Let X be the topological sum of the sequence S_n, n= 1,2,3,... and $\{x_n\} \cup \{x\}$. Since each S_n, n= 1,2,3,... and$\{X_n\} \cup \{X\}$ is s-sequential, X is s-sequential, by Proposition 1.

Now let Y be a space obtained from X by identifying a_n to x_n.

Let f: $X \to Y$ be the natural mapping. By Theorem 4 in [25], Y is a s-sequential space but not a s-Frechet Urysohn space.

Let W be a weak base consists of the following three types of collection of weak neighbourhood of x'

If $x' \neq x$ then

$$T_{x'} = \begin{cases} \{x'\}, & \text{if } x' \in S_n \setminus \{x_n\} \\ \text{Basis of } S_n, & \text{if } x' = x_n, \text{for some } n \end{cases}$$

If x'=x, $U \in T_x$ then implies that

$$U = \{x_n : n \in N' \text{and } d(N') = 1\} \cup \{S_n \setminus S_n' : S_n' \text{ is a non-thin subsequence } S_n \text{ when } n \in N'\} \cup \{x\}.$$

Then W is a weak base of Y. But for all $U \in T_{x,}$ $x \notin IntU$ Therefore, Y is not a s-Frechet Urysohn space by Theorem 4 in [25]. Now let $Y' = Y \setminus \{xn / n \in \mathbb{N}\}$.

Then $\{x\}$is s-sequentially open in Y'. Suppose$\{x\}$is not a s-sequentially open. Then there is a sequence $S = \{x_n'\}$ in Y' s-converges to x. Then we get a contradiction to Y is not a s-Frechet Urysohn space.

Let A be a subset of Y with $x \in \overline{A}$.

If $x_n \in A$, for all $n \in N'$ where N' is a non-thin subsequence of$\{X_n\}_{n \in N}$ then $\{X_n\}_{n \in N'}$ s-converges to x, by Remark 1.2 (3) in [25].

If $x_n \in A$, for all $n \in N'$ where N' is a thin subsequence of $\{X_n\}_{n \in N}$ Then $A \cap S_n$ is a non-thin subsequence of S_n for all $n \in N''$ where N'' is a non-thin subsequence of $\mathbb{N}$

Suppose $A \cap S$ is a non-thin subsequence of S, then $A \cap S$ itself is a s-convergent sequence.

Suppose $A \cap S$ is a thin subsequence of S, then S\A is again s-convergent to x. Now we form a one-to-one function f between $\mathbb{N} \setminus \mathbb{N}''$ and N'' by n^{th} element in $\mathbb{N} \setminus \mathbb{N}''$ goes to n^{th} element in N′ Do the same between S\A and $S'(=A)$ by $x_1' = x_{n,m}$ goes to $x_1'' = x_{f'(n,m)}$ that is, m^{th} element of $A \cap S_{f(n)}$ Then $\{x''\}$ is a sequence in A s-converges to x. Suppose not $\{n : x_n'' \notin U\}$ is a non-thin subsequence of $\mathbb{N}$. Now we form an open set U' from U as follows:

If $S_n \cap U$, is non-dense in S_n where $n \in N''$, then eliminate the elements of S_n from U to get U'.

If $S_n \cap U$ is dense in S_n where $n \in N\mathbb{N}''$ and S_n (S\A) is a non-thin subsequence of S_n, then eliminate the element $x_{n,m}$ from U to get U' where $x_{f'(n,m)} \notin U$. Then $U' \cap S_n$ is dense in $S_{n,}$ since both $S_n \cap$ (S\A), and $S_{f(n)} \cap A$ are non-thin subsequences of S_n and $S_{f(n)}$ respectively.

Now $\{n : x_n'' \notin U\} \subset \{n : x_n' \notin U'\}$ which is a non-thin subsequence of $\mathbb{N}$. Therefore, $S \setminus A$ is not a s-convergent sequence which is a contradiction.

Therefore, $\{x_n''\}$ is a sequence in A s-convergent to x and hence Y is s-Frechet Urysohn space which is a contradiction. Therefore, $\{x\}$ is s-sequentially open in Y' but not open in Y'. Therefore, Y' is not s-sequential.

Next we see the necessary and sufficient condition for which a subspace of an most general case I-sequential space is I-sequential.

Theorem 5

A subspace Y of an I-sequential space X is I-sequential iff $\varphi_X |_{\varphi_X^{-1}(Y)}$ is a quotient.

Proof. Let $Y' = \varphi_X^{-1}(Y)$ and $\varphi' = \varphi_X |_{Y'}$ and let $\varphi_Y : Y^* \to Y$ be a mapping as in the proof of Proposition 3. Suppose ϕ' is a quotient mapping and let $\varphi_Y^{-1}(U)$ be open in $Y^*$where $U \subset Y$.

Since ϕ' is quotient, it is enough if we prove that $\phi'^{-1}(U)$ is open in Y'.

Let $(x,S) \in \phi'^{-1}(U)$ where S=$\{x_n\} \cap \{x\}$. Suppose $(x,S) \in \phi'^{-1}(U) \subset \phi'^{-1}(U)$, then there is nothing to prove.

Suppose $(x,S) \notin \phi'^{-1}(U)$ and $(x,S) \in \phi'^{-1}(U)$

Let U′=$\{(x,S)$: $x_n Y\} \cup \{(x,S)\}$. Since $I(S,x)$, is open in X^* and $\varphi_X^{-1}(Y)$ is a subspace of X,* $I(S,x) \cap \varphi_X^{-1}(Y) = U'$ is open in $\varphi_X^{-1}(Y)$.

Now let

$$y_n = \begin{cases} x_n, & \text{if } x_n \in Y \\ x, & \text{if } x_n \notin Y \end{cases}$$

Then $\{Y_n\}$in Y I-converges to $x \in U$ which implies (Y_n,S') I-converges to $(x,S') \in \varphi_Y^{-1}(U)$. That is

$$I = \{n : (x_n, S') \notin \varphi_Y^{-1}(U)\}$$

$$I = \{n : (x_n, S') \notin \varphi_Y^{-1}(U)\}$$

$\{n{:}x_n \notin U \text{ and } x_n \in Y\}$

$\{n{:}(x_n,S) \notin \phi'^{-1}(U) \text{ and } (x_n,S) \in \varphi_X^{-1}(Y)\}$

Now let $V=\{(x_n,S){:}n\in I\}$. Then $I(S,x)\setminus V$ is open in $I(S,x)$. Therefore, $I(S,x)\setminus V \cap \varphi_X^{-1}(Y) = U'\setminus V$ is open in $\phi'^{-1}(Y)$ and $(x,S)\in U'\setminus V\subset\phi'^{-1}(U)$ Therefore, $\phi'^{-1}(U)$ is open in Y' and hence U is open in Y. Therefore, ϕ_y is a quotient mapping. By Theorem 2 and Proposition 1 in [1], Y is an I-sequential space.

Conversely, let Y be an I-sequential space. By Proposition 3, ϕ_y is a quotient mapping. Now let $\phi'^{-1}(U)$ is open in Y' where $U\subset Y$ Since $\phi'^{-1}(U)= \phi'^{-1}(U)\cap Y^*$ is open in Y^*, U is open in Y. Therefore, ϕ' is a quotient mapping [26,27].

Finally, we give some easy propositions whose proofs are omitted.

Proposition 5

The continuous open or closed image of a I-sequential space is I-sequential.

Proposition 6

If the product space is I-sequential, so is each of its factors.

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