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Stability of the Stationary Solutions in the Bounded Problem of Eight Bodies with Incomplete Symmetry

Elena Cebotaru*

Introduction

It is investigated the stability problem in the Liapunov sense of a new class of exact solutions of the bounded and flat Newtonian problem of several bodies with incompleted symmetry. Whether in the non-coordinate space P_0xyz there is the movement of eight bodies P_0, P_1, P_2, P_3, P_4, P_5, P_6, P, each having the masses m_0, m_1, m_2, m_3, m_4, m_5, m_6, μ, which attract each other in accordance with the law of the universal attraction. We will study the planar dynamic pattern formed by a square in the vertices of which the points P_1, P_2, P_3, P_4 are located, the other two points P_5, P_6, having the masses $m_5=m_6$, are on the diagonal P_1P_3 of the square at the distances equal to the point P_0, in around which this configuration rotates at a constant speed ω determined precisely by the model parameters. The motion of the infinitely small mass $\mu=0$ (the so-called passive gravitational body) will be studied in the gravitational field formed by the seven bodies P_0, P_1, P_2, P_3, P_4, P_5, P_6 that attract each other and attract the body P.

The Liapunov sense of a new class of exact solutions of the restricted and flat Newtonian problem of several bodies with incomplete symmetry is investigated. In the studied model $m_7=\mu=0$. For simplicity it will be considered $P(x_7,y_7,z_7)\equiv P(x,y,z=0)$ further and then the equations of the point $P(x,y,z=0)$ movement have the form (Figure 1):

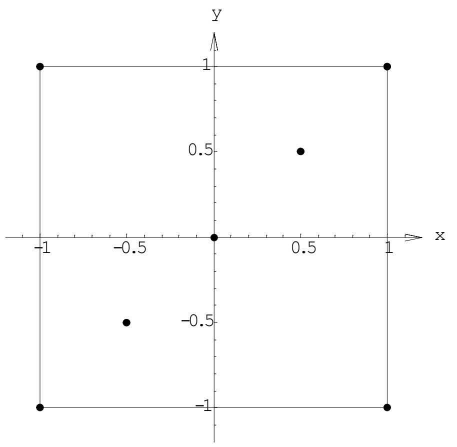

Figure 1: Studied model of incomplete symmetry.

***Corresponding author:** Elena Cebotaru, Stefan cel Mare si Sfant Boulevard 168, Chisinau 2004, Moldova, Tel: +373 22 237 861; E-mail: elena.cebotaru@mate.utm.md

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$$\begin{cases} \dfrac{d^2x}{dt^2}+\dfrac{fm_0x}{r^3}=\dfrac{\partial R}{\partial x}, \\ \dfrac{d^2y}{dt^2}+\dfrac{fm_0y}{r^3}=\dfrac{\partial R}{\partial y}, \end{cases} \tag{1}$$

where

$$\begin{cases} R=f\sum_{j=1}^{6}m_j\left(\dfrac{1}{\Delta_{kj}}-\dfrac{xx_j+yy_j}{r_j^3}\right), \\ \Delta_j^2=\left(x_j-x\right)^2+\left(y_j-y\right)^2, \\ r_j^2=x_j^2+y_j^2, r^2=x^2+y^2. \end{cases} \tag{2}$$

To determine ω them, we will perform such a coordinate transformation that would exclude from the right the equations describing the motion of the bodies during t [1-4]:

$$\text{To deter}\begin{cases} x_j=X_j\cos(\omega t)-Y_j\sin(\omega t), \\ y_j=X_j\sin(\omega t)+Y_j\cos(\omega t). \end{cases} \tag{3}$$

Let $P_1(1,1)$, $P_2(-1,1)$, $P_3(-1,-1)$, $P_4(1,-1)$, $P_5(\alpha,\alpha)$, $P_6(-\alpha,-\alpha)$, $f=1$, $m_0=1$, $m_5=m_6$ then, applying the symbolic calculus system Mathematica (SCS Mathematica), we obtain:

$$m_1=m_3,\ m_2=m_4=f_1(\alpha,m_1),\ m_5=m_6=f_1(\alpha,m_1),\omega^2=f_3(\alpha,m_1) \tag{4}$$

The following table shows the tolerable intervals of α it depending on the values of m_1 its calculated approximately using the graphical tools of SSC Mathematics (Table 1):

$$\begin{cases} u=0, v=0, \\ \omega^2x+2\omega v-\dfrac{fm_0x}{r^3}+\dfrac{\partial R}{\partial x}=0, \\ \omega^2y\ -2\omega u-\dfrac{fm_0y}{r^3}+\dfrac{\partial R}{\partial y}=0, \end{cases} \tag{5}$$

To determine them, the graphical possibilities of SSC Mathematics were used:To determine them, the graphical possibilities of SCS Mathematica were used:

We will note, for simplicity, the coordinates of any point Ni, Si through $x_i^*, y_i^*, z_i^*=0$ and through the vector

Table 1: According to the definition of the stationary solutions of differential equations, the equilibrium positions (if they exist) are the solutions of the functional equation system.

m_1	**Admissible intervals for α**
0.0001	---------------
0.001	---------------
0.01	(0.8582; 0.85857)
0.1	(0.715; 0.718)
1	(0.48965; 0.5053)
10	(0.291; 0.320)
100	(0.149; 0.2871)
1000	(0.050; 0.2838)

$$x=(u-u^*,\ v-v^*,\ w-w^*,\ x-x^*,\ y-y^*,\ z-z^*) \tag{6}$$

The six-dimensional phase space $\{x\}$ is local, therefore each of the N_i and S_i equilibrium points (taken separately) represents the point $x=0$ of this space. By performing the linearization procedure in the vicinity of the phase point with SCS Mathematica we obtain the following system of linear differential equations (Table 2):

$$\frac{dx}{dt}=Ax, \tag{7}$$

Where the matrix A of the size 6x6 has the form:

$$A=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a & b & 0 & 0 & 2\omega & 0 \\ b & c & 0 & -2\omega & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

For each equilibrium position the values of the elements *a, b, c, d* of the matrix *A* will be different. The characteristic equation from which the matrix A's own values are determined is:

$$det\,(A-\lambda E)=(\lambda^2-d)\,(\lambda^4+(4\omega^2-a-c)\,\lambda^2+ac-b^2)=0 \tag{9}$$

In order for each of the researched equilibrium positions to be stable, it is necessary that all solutions of the equation to be imaginary. As $d<0$ we obtain that two matrix values *A* will always be imaginary. We will note them in the future through λ_5, λ_6 Table 3.

Theorem 1

There are values of the parameters m_1 and α for which the bisectoral stationary points S_i of the boundary problem of eight bodies are stable in the first approximation.

Normalization of the Hamiltonian's square part

The study of the stability in the Liapunov sense of the stationary points in the fourth order Hamiltonian systems can be made only on the basis of Arnolid-Mozer theorem. In order to verify that the conditions of this theorem are met in the previously studied model, we will first attempt to bring Birkg of into normal form in a series of powers in the vicinity of any stationary point, stable in the first aroximation. In the subsequent calculations and transformations, the stable stationary point was used in the first approximation S_1 with the coordinates [5-7].

$$x^*=1.4116760833927924,\ y^*=-0.12379179384743404, \tag{10}$$

obtained for $m_1=0.01$, $\alpha=0.8584$, We build, in a rather small neighborhood of this point, the decomposition of Hamiltonian's series of powers with the accuracy up to the fourth power of the *X, Y* coordinates and the P_X, P_Y impulses. We will have:

$$H=H_2(X,Y,P_X,P_Y)+H_3(X,Y)+H_4(X,Y)+R_5(X,Y), \tag{11}$$

Where H_k,(k=2,3,4) there is a homogeneous k-grade, and the R_5 rest of the decompose in the Taylor series. For the studied case the square shape and the 3 and 4 forms are equal to:

$$H_2=0.5\left(-0.68942X^2+0.32466Y^2+P_X^2+P_Y^2+0.17922XY+1.19431\left(YP_X-XP_Y\right)\right) \tag{12}$$

$$H_3=0.1667(1.4248X^3-0.7599X^2Y-2.007XY^2+0.1931Y^3) \tag{13}$$

$$H_4=0.04167(-4.01396X^4+3.7127X^3Y+11.6388X^2Y^2-2.8827XY^3-1.5419Y^4) \tag{14}$$

Table 2: Possibilities of SCS Mathematica.

m_1	α	N_1		S_1	
		x^*	y^*	x^*	y^*
0.01	0.8583	1.15589	1.15589	1.39868	-0.22286
0.01	0.8584	1.15597	1.15597	1.41168	-0.12379
0.01	0.8585	1.15604	1.15604	1.41684	-0.05223
0.01	0.85853	1.15606	1.15606	1.41760	-0.03417
0.1	0.715	1.34188	1.34188	1.34865	-0.45766
0.1	0.717	1.34324	1.34324	1.44139	-0.11335
1	0.48965	1.63351	1.63351	0.93934	-1.05917
1	0.505	1.66022	1.66022	1.82285	-0.00771
10	0.291	1.84521	1.84521	2.19692	-0.00052
100	0.2	1.82945	1.82945	0.82914	-0.02594
1000	0.2	1.81083	1.81083	2.10424	-0.05038

Table 3: Values for stationary points N_i and S_i.

m_1	α	N_1		S_1	
		λ_1,λ_2	λ_3,λ_4	λ_1,λ_2	λ_3,λ_4
0.01	0.8583	±1.30918	±1.12374*i*	±0.28434*i*	±0.51826*i*
0.01	0.8584	±1.30792	±1.12295*i*	±0.49471*i*	±0.32201*i*
0.01	0.8585	±1.30666	±1.12216*i*	±0.45941*i*	±0.36935*i*
0.01	0.85853	±1.30627	±1.12197*i*	±0.00440*i*+0.36926*i*	±0.00440-0.36926*i*
0.1	0.715	±1.19131	±1.06789*i*	±0.34443+0.53193*i*	±0.34443-0.53193*i*
0.1	0.717	±1.17894	±1.06051*i*	±0.40784+0.56449*i*	±0.40784-0.56449*i*
1	0.48965	±1.36716	±1.30616*i*	±0.74472+0.82809*i*	±0.74472-0.82809*i*
1	0.505	±1.23329	±1.12811*i*	±0.75807+0.83104*i*	±0.75807-0.83104*i*
10	0.291	±2.50383	±2.63038*i*	±1.6617+1.88497*i*	±1.6617-1.88497*i*
100	0.2	±8.22619	±8.56881*i*	±15.3124	±8.390991*i*
1000	0.2	±27.1564	±28.0709*i*	±17.7615+19.8928*i*	±17.7615-19.8928*i*

Berkgof's theorem on Hamiltonian normalization indicates that first such an un-generated transformation $(X,Y,P_X,P_Y) \to (p_1, p_2, q_1, q_2)$ should be found that would exclude from H_2 the square form the products of impulses and coordinates $(p_1q_1, p_2q_1, q_1q_2, q_2p_1, q_2p_2, p_1p_2)$ and leave only their squares $p_1^2, p_2^2, q_1^2, q_2^2$. In addition, the coefficient of the sum $\left(p_1^2+q_1^2\right)$ must be the size $\frac{\sigma_1}{2}=\frac{|\lambda_1|}{2}$, and the sum $\left(p_2^2+q_2^2\right)$ - the size $-\frac{\sigma_2}{2}=-\frac{|\lambda_3|}{2}$, where the λ_1, λ_3 different values of the matrix are different A. We will look for these transformations in the form of:

$$\begin{bmatrix} X \\ Y \\ P_X \\ P_Y \end{bmatrix} = B_4 \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix}, \tag{15}$$

Where B_4 is an unknown matrix of size 4×4. The matrix elements, after performing the necessary matrix transformations, are determined from the system of linear equations of the order of 16:

$$C_{16} z = 0 \tag{16}$$

Where $Z^T = (b_{11}, b_{12}, \ldots, b_{44})$ the vector transposed by the size 16 is made up of the matrix elements

$$B_4 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}. \tag{17}$$

C_{16} is a matrix of size with known elements. For point S_1, a simplex matrix B_4 exists and is equal to:

$$B_4 = \begin{pmatrix} 0.655878 & -0.96524 & -1.88012 & -1.43332 \\ 4.22743 & -4.70206 & 0.7917 & 1.79003 \\ -1.59432 & 2.34632 & -0.148299 & -0.758111 \\ 0 & 0 & 0.968621 & 0.658175 \end{pmatrix}. \tag{18}$$

By making the corresponding transformations the forms K_2, K_3, K_4 of Hamiltonian K (written in the new coordinates) will be determined from the relations:

$$K_2 = \frac{1}{2}\sigma_1\left(p_1^2+q_1^2\right) - \frac{1}{2}\sigma_2\left(p_2^2+q_2^2\right) = 0.247354\left(p_1^2+q_1^2\right) - 0.161002\left(p_2^2+q_2^2\right),$$

$$\begin{aligned} K_3 = &-1.52247p_1^3 - 2.75988p_1^2p_2 - 0.560614p_1p_2^2 + 0.555805p_2^3 + \\ &+4.33427p_1^2q_1 + 13.6424p_1p_2q_1 + 8.14701p_2^2q_1 + 11.8376p_1q_1^2 + \\ &8.80617p_2q_1^2 - 1.65267q_1^3 - 5.45395p_1^2q_2 - 16.1243p_1p_2q_2 - \\ &-9.30725p_2^2q_2 - 25.8349p_1q_1q_2 - 18.3724p_2q_1q_2 + 7.0171q_1^2q_2 + \\ &+14.0106p_1q_2^2 + 9.47119p_2q_2^2 - 9.47983q_1q_2^2 + 4.134q_2^3, \end{aligned}$$

$$\begin{aligned} K_4 = &-1.74247p_1^4 - 2.96036p_1^3p_2 + 2.1337p_1^2p_2^2 + 5.36404p_1p_2^3 + \\ &+1.99884p_2^4 + 11.3635p_1^3q_1 + 44.9959p_1^2p_2q_1 + 47.2451p_1p_2^2q_1 + \\ &+12.9554p_2^3q_1 + 30.1269p_1^2q_1^2 + 33.4298p_1p_2q_1^2 - 2.83397p_2^2q_1^2 - \\ &-22.8284p_1q_1^3 - 43.3045p_2q_1^3 - 22.588q_1^4 - 13.7101p_1^3q_2 - \\ &-52.0847p_1^2p_2q_2 - 53.0885p_1p_2^2q_2 - 14.0642p_2^3q_2 - 64.3346p_1^2q_1q_2 - \\ &-65.2487p_1p_2q_1q_2 + 11.9894p_2^2q_1q_2 + 84.7749p_1q_1^2q_2 + \\ &+151.297p_2q_1^2q_2 + 99.8009q_1^3q_2 + 34.0675p_1^2q_2^2 + 30.8765p_1p_2q_2^2 - \\ &-9.92384p_2^2q_2^2 - 103.733p_1q_1q_2^2 - 175.458p_2q_1q_2^2 - 164.82q_1^2q_2^2 + \\ &+41.8947p_1q_2^3 + 67.5526p_2q_2^3 + 120.569q_1q_2^3 - 32.9581q_2^4. \end{aligned}$$

Normalization in Berkgof sense of the cubic form and Hamiltonian's fourth order form

In order to move to the angles-of-action variables, we will use Berkgof's classic transformation:

$$\begin{cases} q_1 = \sqrt{2\tau_1}\sin\theta_1, q_2 = \sqrt{2\tau_2}\sin\theta_2, \\ p_1 = \sqrt{2\tau_1}\cos\theta_1, p_2 = \sqrt{2\tau_2}\cos\theta_2, \end{cases} \tag{19}$$

where the new variables τ_1, τ_2, θ_1, θ_2 are angular-action variables. If we write the new Hamiltonian F in the form:

$$F(\theta_1, \theta_2, \tau_1, \tau_2) = F_2(\tau_1, \tau_2) + F_3(\theta_1, \theta_2, \tau_1, \tau_2) + F_4(\theta_1, \theta_2, \tau_1, \tau_2) + \ldots, \tag{20}$$

then after performing the corresponding transformations we obtain:

$$F_2(\tau_1, \tau_2) = \sigma_1\tau_1 - \sigma_2\tau_2 = 0.494707884724482 07\tau_1 - 0.3220047802085036\tau_2 \tag{21}$$

In the vicinity of the stationary point the Hamiltonian equations in the new coordinates are expressed by the formulas:

$$\begin{cases} \frac{d\theta_1}{dt} = \frac{\partial F_2}{\partial \tau_1} + \frac{\partial F_3}{\partial \tau_1} + \frac{\partial F_4}{\partial \tau_1} + \ldots, \frac{d\theta_2}{dt} = \frac{\partial F_2}{\partial \tau_2} + \frac{\partial F_3}{\partial \tau_2} + \frac{\partial F_4}{\partial \tau_2} + \ldots, \\ \frac{d\tau_1}{dt} = -\frac{\partial F_3}{\partial \theta_1} - \frac{\partial F_4}{\partial \theta_1} + \ldots, \frac{d\tau_2}{dt} = -\frac{\partial F_3}{\partial \theta_2} - \frac{\partial F_4}{\partial \theta_2} + \ldots, \end{cases} \tag{22}$$

The Arnolid-Mozer theorem requires the construction of yet another canonical transformation

$$(\theta_1, \theta_2, \tau_1, \tau_2) \to (\psi_1, \psi_2, T_1, T_2) \tag{23}$$

which would nullify the third order shape in the Hamiltonian transfomation, and would exclude from the shape of the four phased angles, yet leaving the corresponding square shape $F_2(\tau_1, \tau_2)$ unchanged.

We will look for this transformation into form:

$$\begin{cases} \theta_1 = \psi_1 + V_{13}(\psi_1, \psi_2, T_1, T_2) + V_{14}(\psi_1, \psi_2, T_1, T_2), \\ \theta_2 = \psi_2 + V_{23}(\psi_1, \psi_2, T_1, T_2) + V_{24}(\psi_1, \psi_2, T_1, T_2), \\ \tau_1 = T_1 + U_{13}(\psi_1, \psi_2, T_1, T_2) + U_{14}(\psi_1, \psi_2, T_1, T_2), \\ \tau_2 = T_2 + U_{23}(\psi_1, \psi_2, T_1, T_2) + U_{24}(\psi_1, \psi_2, T_1, T_2), \end{cases} \tag{24}$$

where unknown functions $V_{13}, V_{23}, U_{13}, U_{23}$ are third order shapes, and $V_{14}, V_{24}, U_{14}, U_{24}$ are fourth order forms with respect to T_1, T_2.

Performing the transformation (22) by means of the determined functions $V_{13}, V_{23}, U_{13}, U_{23}, V_{14}, V_{24}, U_{14}, U_{24}$ is obtained for the Hamiltonian W in the vicinity of the stationary point S_1 with the coordinates (10), calculated for the $m_1=0.01$ $\alpha=0.8584$, final form:

$W(\psi_1, \psi_2, T_1, T_2) = W_2(T_1, T_2) + W_4(T_1, T_2) + F_5(\psi_1, \psi_2, T_1, T_2) + \ldots$, where

$$W_2(T_1, T_2) = \sigma_1 T_1 - \sigma_2 T_2 = 0.49470788472448207T_1 - 0.32200478020850365T_2 \tag{25}$$

$$\begin{gathered} W_4(T_1, T_2) = c_{20}T_1^2 + c_{11}T_1T_2 + c_{02}T_2^2, \\ c_{20} = -41.5987, c_{11} = -458.902, c_{02} = 64.1789. \\ W_4(\sigma_1, \sigma_2) = 65.918 \neq 0. \end{gathered} \tag{26}$$

Thus, the purpose of all the above transformations consisted in the fact that after their execution the square W_2 and the fourth order W_4 depend only on the impulses T_1, T_2, and the cubic form is canceled. $(W_3 \equiv 0)$.

Similar results were obtained for the other equilibrium bisectorial positions S_i. This result indicates that all the calculations made in

SCS Mathematica are correct and consistent with the theoretical conclusions resulting from the symmetry of the studied gravitational model. Thus it can be concluded that stable stationary points in the first approximation are also stable in Liapunov sense [8-12].

Theorem 2

There are values of the parameter $m_I$and corresponding values of the parameter for α which the stationary points of the boundary problem of the eight bodies are stable not only in the first approximation but are also stable in the Liapunov sense.

Conclusion

There are values of the parameters m_I and α for which the bisectoral stationary points S_i of the boundary problem of eight bodies are stable in the first approximation and in the Liapunov sense.

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Author Affiliation **Top**

Technical University of Moldova, Moldova