Upper Semicontinuous Representability of Maximal Elements for Non total Preorders on Compact Spaces

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Abstract
We discuss the possibility of determining all the maximal elements of a preorder on a compact topological space by maximizing all the functions in a suitable family of upper semicontinuous order-preserving functions.

Keywords
Preorder; Order-preserving function; Weak utility; Maximal element; Upper semicontinuous function

Introduction
White’s theorem [1] is important since, for every maximal element \( x \) relative to a preorder \( \preceq \) on set \( X \), it guarantees the existence of an order-preserving function \( u \) on the preordered set \((X, \preceq)\) attaining its maximum at \( x \). Given a preordered set \((X, \preceq)\), a point \( x \in X \) is said to be a maximal element for \( \preceq \) if for no \( z \in X \) it occurs that \( x \preceq z \). In the sequel we shall denote by \( X^M_\preceq \) the set of all the maximal elements of a preordered set \((X, \preceq)\). Please observe that \( X^M_\preceq \) can be empty.

In this paper, we generalize White’s theorem to the upper semicontinuous case. This means that we present conditions on a preorder \( \preceq \) on a topological space \((X, \tau)\) under which, for every maximal element \( x \), there exists an upper semicontinuous order-preserving function \( u \) on the preordered topological space \((X, \tau, \preceq)\) attaining its maximum at \( x \) provided that an upper semicontinuous order-preserving function \( u \) on \((X, \tau, \preceq)\) exists. It should be noted that Bevilacqua et al. [3] already characterized the property according to which every maximal element relative to a preorder on a compact topological space can be obtained by maximizing a transfer weakly upper continuous weak utility for its strict part (see the generalization of Weierstrass Theorem presented by Tian et al. [4]).

It is clear that these results are important due to the well-known fact that every upper semicontinuous (more generally transfer weakly upper continuous) function attains its maximum on a compact topological space, and the nearly obvious consideration that a point \( x \), at which an order-preserving function \( u \) for a preorder (or, more generally, a weak utility for its strict part) attains its maximum is also a maximal element for \( \preceq \).

Notation and preliminaries
Let \( X \) be a nonempty set (decision space). A binary relation \( \preceq \) on \( X \) is interpreted as a weak preference relation, and therefore, for any two elements \( x, y \in X \), the scripture ”\( x \preceq y \)” has to be thought of as ”the alternative \( y \in X \) is at least as preferable as \( x \in X \)”. As usual, \( \prec \) denotes the strict part of a binary relation \( \preceq \) (i.e., for all \( x, y \in X \), \( x \prec y \) if and only if \((x \preceq y) \) and not \((y \preceq x) \)). A preorder is a reflexive and transitive binary relation. An anti-symmetric preorder \( \preceq^\sim \) is referred to as an order. Furthermore, \( \sim \) stands for the indifference relation (i.e., for all \( x, y \in X \), \( x \sim y \) if and only if \((x \preceq y) \) and \((y \preceq x) \)). We have that \( \sim \) is an equivalence relation on \( X \) whenever \( \preceq^\sim \) is a preorder.

For every \( x \in X \), we set \( l(x)=\{z \in X: z \prec x\} \) and \( i(x)=\{z \in X: z \sim x\} \).

Given a preordered set \((X, \preceq)\), a point \( x \in X \) is said to be a maximal element for \( \preceq \) if for no \( z \in X \) it occurs that \( x \preceq z \). In the sequel we shall denote by \( X^M_\preceq \) the set of all the maximal elements of a preordered set \((X, \preceq)\). Please observe that \( X^M_\preceq \) can be empty.

We recall that a function \( u: (X, \preceq) \rightarrow (R, \leq) \) is said to be

i. isotonic or increasing if, for all \( x, y \in X \), \( x \preceq y \Rightarrow u(x) \leq u(y) \);
ii. a weak utility for \( \prec \) if, for all \( x, y \in X \), \( x \prec y \Rightarrow u(x) < u(y) \);
iii. strictly isotonic or order-preserving if it is both isotonic and a weak utility for \( \sim \).

Strictly isotonic functions on \((X, \preceq^\sim)\) are also called Richter-Peleg representations of \( \preceq^\sim \) in the economic literature (see e.g. Richter et al. [5] and Peleg et al. [6]).

A preorder \( \preceq \) on a topological space \((X, \tau)\) is said to be

i. upper semicontinuous if, for all \( x, y \in X \), \( i(x)=\{z \in X: x \preceq^\sim z\} \) is a closed subset of \( X \) for every \( x \in X \);
ii. Quasi upper semicontinuous if there exists an upper semicontinuous preorder \( \preceq^\sim \) on \((X, \tau)\) such that \( \prec \subset \prec^\sim \).

An upper semicontinuous preorder Ward et al. [7] or more generally a quasi-upper semicontinuous preorder Bosi et al. [8, Theorem 3.1] \( \preceq \) on a compact topological space \((X, \tau)\) admits a maximal element. As usual, for a real-valued function \( u \) on \( X \), we denote by \( \arg \max u \) the set of all the points \( x \in X \) such that \( u \) attains its maximum at \( x \) (i.e., \( \arg \max u=\{z \in X: u(z)=\max_u(X)\}\)).

If \( \tau \) is a topology on a set \( X \), and \( \preceq \) is a preorder on \( X \), then the triplet \((X, \tau, \preceq)\) will be referred to as a topological preordered space. The (natural) (interval) topology on the real line \( R \) will be denoted by \( \tau_{NI} \).

Finally, we recall that a real-valued function \( u \) on a topological space \((X, \tau)\) is said to be upper semicontinuous if \( u([a, \infty))=\{x \in X: \)

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$u(x) < a$ is an open set for all $a \in \mathbb{R}$. A very well know result guarantees that every upper semicontinuous real-valued function $u$ on a compact topological space $(X, \tau)$ attains its maximum.

**Maximal elements of preorders from maximization of upper semicontinuous functions**

The following theorem was proved by White et al. [1]. Given any maximal element $x_\circ$ relative to a preorder $\preceq$ on a set $X$, it guarantees the existence of some order-preserving function $u$ attaining its maximum at $x_\circ$, provided that an order-preserving function $u : (X, \preceq) \to (\mathbb{R}, \leq)$ exists. Therefore, in order to determine all the maximal elements of a preorder $\preceq$ on a set $X$, the agent maximizes all the functions $u$ in a family $U$ of bounded order-preserving functions for $\preceq$. Needless to say, this is a very important opportunity, at least theoretically.

**Theorem:** (White et al. [1]): Let $(X, \preceq)$ be a preorder set and assume that there exists an order-preserving function $u : (X, \preceq) \to (\mathbb{R}, \leq)$. If $X_\circ$ is nonempty, then for every $x_\circ \in X_\circ$ there exists an order-preserving function $u : (X, \preceq) \to (\mathbb{R}, \leq)$ such that $\arg \max u = (z \in X : z \prec x)$. Then the following conditions are equivalent:

(i) There exists an upper semicontinuous order-preserving function $u : (X, \preceq) \to (\mathbb{R}, \leq)$ such that $\arg \max u = (z \in X : z \prec x)$ is a closed subset of $X$.

(ii) There exists an upper semicontinuous order-preserving function $u : (X, \preceq) \to (\mathbb{R}, \leq)$ such that $\arg \max u = (z \in X : z \prec x)$ is a closed subset of $X$.

**Proof:**

(i) $\Rightarrow$ (ii) Let $\preceq$ be an upper semicontinuous order-preserving function on $(X, \tau, X)$. Without loss of generality, we can assume $x_\circ$ to be bounded. Consider a point $x_\circ \in X_\circ$ and define the real-valued function $f$ on $X$ as follows for any choice of a positive real $\delta$:

$u(x) = \{ \begin{cases} \text{not } (x \prec x_\circ) \\ u(x) + \delta \text{ if } (x \prec x_\circ) \end{cases}$

White et al. [1, Theorem 1] proved that the above function $u$ is order-preserving for $\preceq$ as soon as $u$ is order-preserving for $\preceq$. For the sake of completeness, let us recall here the arguments supporting this consideration. It is clear that $u^{-1}(x) \leq u(x)$ for every $x \in X$. In order to show that $u$ is increasing with respect to $\preceq$, consider any two points $x, y \in X$ such that $x \preceq y$. Since $\preceq$ is a preorder, then it is clear that $u(x) \leq u(y)$ from the definition of $u$. On the other hand, if not($y \preceq x$), then it must also be not($x \prec y$), and in turn $x \prec y$ due to the fact that $x_\circ$ is a maximal element relative to $\preceq$. Hence, since $x \prec x_\circ$, $y \preceq x$, and $x \prec y$, we have that $u(x) = u(x) \leq u(y) = u(x)$ from the definition of $u$ and the fact that $u$ is increasing with respect to $\preceq$. In order to show that $u$ is a weak utility for $\preceq$, consider any two points $x, y \in X$ such that $x \prec y$. Then we have that not($x \prec x_\circ$), since $x_\circ \prec x \prec y$ implies that $x_\circ \prec y$ (a contradiction, since $x_\circ$ is assumed to be a maximal element for $\preceq$). Therefore, from the definition of $u$ and the fact that $\preceq$ is a weak utility for $\preceq$, we have that $u(x) = u(x) \leq u(y)$, which obviously implies that $u(x) = u(y)$. Further, $u$ attains its maximum at $x_\circ$ and actually, since $\delta$ is a positive real, it is clear that (i) $\arg \max u = (z \in X : z \prec x)$.

(ii) $\Rightarrow$ (i) Assume that there exists an upper semicontinuous order-preserving function $u : (X, \tau, \preceq) \to (\mathbb{R}, \leq)$ such that $\arg \max u = (z \in X : z \prec x)$ is not closed, then there exists an element $z \in X$ such that $u(z) \not\in (X : z \prec x)$ for every neighborhood $U$ of the element $z$. This contradicts the fact that $u$ is upper semicontinuous, since in case $u^{-1} \setminus (\infty, u(x))$, an open set containing $x_\circ$ and actually, since $\delta$ is an open set containing $x_\circ$ for which $u(z) = u(x_\circ)$. This consideration completes the proof.

**Remark:** It is clear that Theorem 3.2 generalizes White’s theorem, due to the fact that these two results precisely coincide when we consider the discrete topology $\tau$ on $X$.

Since in order to determine a maximal element relative to preorder $\preceq$ on a set $X$ it suffices to maximize a weak utility for the strict part $\prec$ of $\preceq$, the following corollary can be considered as useful. Indeed, the reader can easily verify that the implication “(i) $\Rightarrow$ (ii)” in Theorem 3.2 is still valid if one considers weak utilities for $\prec$ instead of order-preserving functions $u$ for $\preceq$.

**Corollary:** Let $(X, \tau, \preceq)$ be a topological preordered space with $\tau$ a compact topology. If there exists an upper semicontinuous weak utility $u_\circ$ for $\preceq$, and $[x] = (z \in X : z \prec x)$ is a closed set for all $x \in X_\circ$, then for every $x \in X_\circ$ there exists an upper semicontinuous weak utility $u$ for $\preceq$ such that $\arg \max u = (z \in X : z \prec x)$.

Bosi et al. [8, Theorem 2.11] proved that there exists an upper semicontinuous weak utility for the strict part $\prec$ of a quasi-upper semicontinuous preorder $\preceq$ on a second countable (i.e., with a countable base) topological space $(X, \tau)$. Hence, we get the following corollary.

**Corollary:** Let $(X, \tau, \preceq)$ be a topological preordered space. Assume that $\tau$ is a compact and second countable topology, and that $\preceq$ is quasi upper semicontinuous. If $[x] = (z \in X : z \prec x)$ is a closed set for all $x \in X_\circ$, then for every $x \in X_\circ$ there exists an upper semicontinuous weak utility $u$ for $\preceq$ such that $\arg \max u = (z \in X : z \prec x)$.

**Conclusion**

In this paper, following the spirit of a theorem of White et al. [1], we have presented some results concerning the representation of the set of all maximal elements of a preorder on a compact topological space by means of the maximization of all functions in a suitable family of bounded upper semicontinuous order-preserving functions. The more delicate problem of characterizing the possibility...
of representing all the maximal elements for a preorder on a compact topological space by means of the maximization of finitely many upper semicontinuous order-preserving functions will be hopefully considered in a future paper.

Reference

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