Existence of Solutions for Impulsive Second Order Abstract Functional Neutral Differential Equation with Nonlocal Conditions and State Dependent-Delay

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Abstract
In this paper, we study the existence of mild solutions for the impulsive second order abstract partial neutral differential equations with state dependent delay of the form

$$\frac{d}{dt}[x'(t) + g(t, x_{\rho(t)} \rho(t)) A x(t) + f(t, x_{\rho(t)} \rho(t))] = \Delta x(t) \xi(t), \quad t \neq t_i, \quad t \in [0, a) \quad t \neq t_i.$$

With nonlocal conditions

$$x(0) = x_0 + p(x) \in B$$

$$x'(0) = y_0 + q(x) \in X$$

$$\Delta x(t_i) = I_i x(t_i), \quad t = t_i, \quad i = 1, 2, \ldots, n$$

$$\Delta x'(t_i) = I_i x'(t_i), \quad t = t_i, \quad i = 1, 2, \ldots, n$$

Keywords
Abstract Cauchy problem; Impulsive differential equations; Cosine function; State-dependent delay

Preliminaries

Through this paper, A is the infinitesimal generator of strongly cosine function of bounded linear operators \((C(t))_{t \geq 0}\) on the Banach space \((X, ||\cdot||)\). We denote by \((S(t))_{t \geq 0}\) the associated sine function which is defined by \(S(t)x = \int_0^t \cos(t)ds\) for \(x \in \mathbb{X}\) and \(t \in \mathbb{R}\). In the sequel, \(N\) and \(\mathbb{N}\) are positive constants such that \(||C(t)|| \leq N\) and \(||S(t)|| \leq \mathbb{N}\) for every \(t \in \mathbb{R}\).

In this paper \(|D(A)|\) represents the domain of \(A\) endowed with the graph norm given by \(||x||_A^2 = ||x||^2 + ||Ax||^2||\).

\(x \in \mathbb{D}(A)\) while \(E\) stands for the space formed by the vectors \(x \in \mathbb{X}\) for which \(C(t)x\) is of the class \(C \in \mathbb{R}\). We know from Kisinski \([8-10]\), that \(E\) endowed with the norm \(||x||_E^2 = ||x||^2 + \sup_{t \in [0,a]} ||AS(t)x||\), \(x \in \mathbb{X}\) is a Banach space. The operator-valued function

\[
A = \begin{bmatrix}
C(t) & S(t) \\
AS(t) & C(t)
\end{bmatrix}
\]

is a strongly continuous group of bounded linear operators on the space \(E \times \mathbb{X}\) generated by the operator

\[
A = \begin{bmatrix}
0 & I \\
A & 0
\end{bmatrix}
\]

defined on \(D(A)x \in \mathbb{X}\). It follows from this that \(AS(t)x \in \mathbb{X}\) for each \(x \in \mathbb{X}\).

Furthermore, if \(x: (0, \infty) \rightarrow \mathbb{X}\) is a locally integrable function, then \(z(t) = \int_0^t S(t-s)x(s)ds\) defines an \(E\)-valued continuous function. This is a consequence of the fact that

\[
\int_0^a G(t-s) x(s)ds = \int_0^b G(t-s) \int_0^a C(t-s)x(s)ds ds
\]

defines an abstract phase space \(B\) described axiomatically and \(f\) is \(B \rightarrow \mathbb{X}, g: B \rightarrow \mathbb{X}\).

\(B\chi B \rightarrow \mathbb{X}, \chi: C(\mathbb{X}, \mathbb{X}) \rightarrow B, \chi: C(\mathbb{X}, \mathbb{X}) \rightarrow \mathbb{X}\) and

\[
I, \chi: B \rightarrow \mathbb{X}, i = 1, 2, \ldots, n\]

are appropriate functions and the symbol \(\Delta x(t)\) represents the jump of the function \(x\) at \(t\), which is defined by \(\Delta x(t) = \xi(t) - \xi(t^+).\)

The theory of impulsive differential equations has become an important area of investigation in recent years stimulated by their numerous applications to problems arising in mechanics, electrical, engineering, medicine, biology, ecology etc [1-5].

Neutral functional differential equations with state-dependent delay and non-local conditions appear frequently in applications as model equations and for this reason the study of this type of equations has received great attention. The problem of the existence of solutions for second order functional differential equations with state-dependent delay and also nonlocal conditions have been treated in the literature recently in \([6,7]\). To the best of our knowledge, the existence of solutions the impulsive second order abstract partial neutral functional differential equations with state-dependent delay and also nonlocal conditions is an untreated topic in the literature and this fact is the main motivation of the present work.

Introduction

In this paper, we study the existence of mild solutions for the impulsive second order abstract partial neutral differential equations with state dependent delay of the form

\[
\frac{d}{dt}[x'(t) + g(t, x_{\rho(t)} \rho(t)) A x(t) + f(t, x_{\rho(t)} \rho(t))] = \Delta x(t) \xi(t), \quad t \neq t_i, \quad t \in [0, a) \quad t \neq t_i.
\]

With nonlocal conditions

$$x(0) = x_0 + p(x) \in B$$

$$x'(0) = y_0 + q(x) \in X$$

$$\Delta x(t_i) = I_i x(t_i), \quad t = t_i, \quad i = 1, 2, \ldots, n$$

$$\Delta x'(t_i) = I_i x'(t_i), \quad t = t_i, \quad i = 1, 2, \ldots, n$$

where \(A\) is the infinitesimal generator of a strongly continuous cosine function of bounded linear operator \((C(t))_{t \geq 0}\) defined on a Banach space \((X, ||\cdot||)\), the function \(x: (-\infty, a) \rightarrow X, x(t) = x(t+0)\) belongs to some abstract phase space \(B\) described axiomatically and \(f\) is \(B \rightarrow \mathbb{X}, g: B \rightarrow \mathbb{X}\).

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Received: July 17, 2017 Accepted: January 15, 2018 Published: February 10, 2018
(−∞,0] into X endowed with a semi norm ||·||, and satisfying the following assumptions:

(A1) If x: (−∞,b)→X, b>0, continuous on [0,b] and x∈B, then for every t∈[0,b] the following conditions hold:
   (a) x is in B
   (b) ||xt||≤H||xt||
   (c) ||B(M(t)||∥B+K(t)||sup||∥x(s)||; 0≤≤t|| where H>0 is a constant; K; M; 0→(1,∞), K is continuous, M is bounded and K, M are independent of x.

(A2) For the functions x in (A1), x is in B valued continuous functions on [0,b].

(A3) The space B is complete.

Definition 2.1 (Mild solutions)

A function u: (−∞,a)→X is called a mild solution of the abstract Cauchy problem (1.1) − (1.3) for every x∈D and

\[ u(t)=C(t)x_0+\int_0^t C(t-s)g(s,x_{\rho}(s))ds+\int_0^t S(t-s)\Gamma(s)ds \]

Some of our results is proved using the following well known results.

Theorem 2.2 (Leray Schauder Alternative)[4,pp.61]. Let D be a convex subset of a Banach space X and assume that 0∉D. Let G: D→D be a completely continuous function. Then the map G has a fixed point in D or the set \{x∈D: x=λG(x), 0<λ<1\} is unbounded [11-14].

Theorem 2.3 Sadovskii [15] Let D be a convex, closed and bounded subset of a Banach space X. If F: D→D is a condensing operator, then G has a fixed point in D.

Remark 2.4 The function t→φ(t) is well defined and continuous from the set R(φ)=\{s(t); s(t)∈X, s(t)≤0, t0 to t and there exists a continuous and bounded function \( f: R(φ)×(0,∞) \) such that ||φ||≤∫||f||d||x|| for every t∈D(φ).

Remark 2.5 The condition (2.4) is frequently verified by functions continuous and bounded. In fact, if B verifies axiom Cx in the nomenclature of [12], then there exists L such that ||φ||≤L for every x∈B continuous and bounded function. Consequently, \[ ||φ||≤L sup_{0}^{a} ||f(t)|| \] for every continuous and bounded function \( f: R(φ)×(0,∞) \) and every t0. We also observe that the space CxU(x) verifies axiom Cx in the rest of this paper, M and K are the constants defined by M=sup_{0}^{a}M(t) and K=sup_{0}^{a}K(t).

Using the following lemma for proof of our main result:

Lemma 2.6 [10, Lemma 2.1]

Let x: (−∞,a)→X be a function such that x=ψ and x∈B in PC. Then ||x||≤((M+∫)sup_{0}^{a} ||f(t)||)||x||+K sup_{0}^{a} ||f(t)|| in [0,∞] and \( \int_{0}^{a} ||f(t)|| dt \) where \( M=sup_{0}^{a} ||f(t)|| \) and \( f(t) = \int_{0}^{t} \phi(s)ds \).

The terminology and notations are those general used in functional analysis. In particular, for Banach spaces Z,W, the notation L(Z,W) stands for the Banach space of bounded linear operators from Z into W and we abbreviate this notation to L(Z) when Z=W. Moreover B(x;Z) denotes the closed ball with radius r>0 in Z and for a bounded function x: [a,b]→X and 0≤≤a we employ the notation ||x|| for ||x||=sup||x(s)||;0≤≤s≤≤t||. We briefly state some of the results.

Existence of Solutions

In this section, we establish the existence of mild solutions for the abstract Cauchy problem (1.1) − (1.3). In section 4 some applications are considered.

(H1) The function \( f(x) \) is a continuous function and that the following conditions are verified.

(H2) \( g: X\rightarrow X \) is a continuous function and verifies the following conditions:

(H3) The maps \( I_{i}, \) are continuous each function \( I_{i} \) is completely continuous and there are positive constants \( ||I_{i}|| \leq c_{i}^{*} ||x||^{*} + d_{i}^{*} \) for every \( x \in X \)

(H4) There are positive constants \( P_{i}, Q_{i} \) such that \( ||I_{i}(x) - J_{i}(x)|| \leq P_{i} ||x||^{*} \) and \( ||J_{i}(x) - J_{i}(y)|| \leq Q_{i} ||x||^{*} \) for every \( x, y \in X \)

Theorem 3.1 Assume that the conditions (H1)(−H3) are verified and that \( g(\cdot) \) is completely continuous. Suppose, furthermore that the following conditions hold:

(a) for every 0≤<t≤r and \( r>0 \), the set \( U(t,r)=\{S(r)\phi(x_{t},t): 0≤≤x_{t}, \} \) is compact in \( X \).

(b) \( p(\cdot) \) is completely continuous and there is \( N_{0} \) such that \( ||p(u)||≤N_{0} \) for every \( u \in C(X) \).

(c) for every \( x_{t} \in (t,r) \), \( X_{t} \) is relatively compact in \( X \) and there is \( N_{0} \) such that \( ||p(x)||≤N_{0} \) for every \( x_{t} \in C(X) \).

If \( \mu \geq 1-K_{x}(N_{0} \mu+\sum_{i} d_{i}^{*} ) \) and \( \frac{1}{\mu} \int_{0}^{t} \frac{ds}{W_{i}(s)+W_{i}(t)} \leq \frac{1-K_{x}}{1-\mu^{*}} (N_{0} \mu+\sum_{i} d_{i}^{*}) \) then

\( \int_{0}^{t} \frac{ds}{W_{i}(s)+W_{i}(t)} \leq \frac{1-K_{x}}{1-\mu^{*}} (N_{0} \mu+\sum_{i} d_{i}^{*}) \).
where

\[
C = \frac{1}{1 - \mu} \left( K_u(NH) + NH, r + M + T^\rho \right) + K_u(\bar{N} W) + \bar{N} \eta
\]

\[
+ \sum_{j=1}^N N \lambda_j \frac{d^2}{dt^2} (t) + \bar{N} \lambda_j \frac{d^2}{dt^2} (t)
\]

Then there exists a mild solution of (1.1) – (1.5).

**Proof**

On the space \( (I; X) \) we define the map \( \Gamma: (I; X) \to (I; X) \) by

\[
\Gamma u(t) = C(t; x_0 + p(u)) + S(t; y_0 + q(u)) + g(0, x_0, x_0(t)),
\]

In order to use Leray Schauder alternative and from assumption (A1).

We obtain an a priori bounded for the solution of the integral equation \( u = \beta(t), \lambda \in (0, 1) \) we get,

\[
\left\| \int_0^t \left( C(s; x_0 + p(u)) + S(s; y_0 + q(u)) + g(0, x_0(x_0(s)), x_0'(s)) \right) ds \right\|
\]

\[
+ \sum_{i=1}^N \left( (M + T^\rho) \left\| \int_0^t \left( C(s; x_0 + p(u)) + S(s; y_0 + q(u)) + g(0, x_0(x_0(s)), x_0'(s)) \right) ds \right\| \right)
\]

\[
+ \sum_{i=1}^N \left( (M + T^\rho) \left\| \int_0^t \left( C(s; x_0 + p(u)) + S(s; y_0 + q(u)) + g(0, x_0(x_0(s)), x_0'(s)) \right) ds \right\| \right)
\]

\[
+ \sum_{i=1}^N \left( (M + T^\rho) \left\| \int_0^t \left( C(s; x_0 + p(u)) + S(s; y_0 + q(u)) + g(0, x_0(x_0(s)), x_0'(s)) \right) ds \right\| \right)
\]

Denoting by \( \tilde{\beta}(t) \) right hand of above equation follows that,

\[
\beta(t) = \frac{K_u(NM) + \bar{N} \lambda_j \frac{d^2}{dt^2} (t)}{1 - \mu}
\]

\[
\leq \beta(t) \leq \frac{K_u(NM) + \bar{N} \lambda_j \frac{d^2}{dt^2} (t)}{1 - \mu}
\]

and hence,

\[
\sum_{i=1}^N \left( (M + T^\rho) \left\| \int_0^t \left( C(s; x_0 + p(u)) + S(s; y_0 + q(u)) + g(0, x_0(x_0(s)), x_0'(s)) \right) ds \right\| \right)
\]

Which implies that the set of function \( \{ \beta(t); \lambda \in (0, 1) \} \) is bounded in \( C(I; R) \). This prove that \( \{ U^\lambda(t); \lambda \in (0, 1) \} \) is also bounded in \( C(I; X) \).

Next, we prove that \( \Gamma \) is completely continuous. To this end, we introduce the decomposition \( \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \) where,

\[
\Gamma_1 u(t) = C(t; x_0 + p(u)) + S(t; y_0 + q(u)) + g(0, x_0, x_0(t)),
\]

\[
\Gamma_2 u(t) = -\int_0^t \left( C(s; x_0) + S(s) \right) (t) ds + \int_0^t \left( C(s; x_0) + S(s) \right) (t) ds
\]

\[
\Gamma_3 u(t) = \sum_{0 \leq t < \alpha} C(t; x_0 + p(u)) + \sum_{0 \leq t < \alpha} S(t; y_0 + q(u)) ds
\]

It is easy to show that \( \Gamma_1 \) is completely continuous and that \( \Gamma_3 \) is continuous. Next, by using Ascoli Arezela we prove that \( \Gamma(B(0, C(I; X))) \) is relatively compact \( C(I; X) \).

The set \( \Gamma \) is equicontinuous on \( I \). Let \( \varepsilon > 0 \) and \( \delta > 0 \) such that

\[
| C(s + h) - C(s) | < \varepsilon, \quad | x_0(t) | < \varepsilon, \quad \forall h \in (0, \delta),
\]

\[
| C(t + h) - C(t) | < \varepsilon, \quad | x_0(t) | < \varepsilon, \quad \forall t \in (0, \delta)
\]

For \( u \in B \), and \( | x_0 | < \delta \) with \( t \in I \), we get

\[
\Gamma u(t) = -\int_0^t \left( C(s; x_0) + S(s) \right) (t) ds + \int_0^t \left( C(s; x_0) + S(s) \right) (t) ds
\]

which prove the assertion.

**Step 2**

The set \( \Gamma \) is relatively compact in \( X \) for every \( t \in I \). Let \( \varepsilon > 0 \) and \( \delta > 0 \) such that \( | u(t) | < \delta \), \( | x_0(t) | < \delta \), \( | C(s) | < \delta \), \( | x_0(t) | < \delta \), \( \forall t \in I \), \( \forall u \in B \). From the estimate, \( | u(t, \theta) | < \delta \), \( \forall t \in I \), \( \forall \theta \in [0, 1] \), \( \forall u \in B \) is bounded in \( X \). Using that \( S(t) \) is uniformly Lipschitz on \( I \), we can choose \( 0 \leq s_0 < s_1 < \ldots < s_N = \varepsilon \) such that \( | S(t') - S(t) | < \delta, \forall t, t' \in [0, \varepsilon] \), \( \forall u \in B \).

Bocher integral see [13, lemma 2.1.3] and fact that \( V = (C[I](g(s'), x_0)) \) is relatively compact in \( X \), follows that,

\[
\Gamma u(t) = -\int_0^t \left( C(s; x_0) + S(s) \right) (t) ds + \int_0^t \left( C(s; x_0) + S(s) \right) (t) ds
\]

\[
\in \text{co}(Q) + B_\delta(0, X) + \sum_{i=1}^N (s_i - s_{i-1}) Co(U(t, s_i, \varepsilon))
\]

where \( co(Q) \) denote the convex hull of a set \( Q \). Thus \( \Gamma(B)(t) \) is relatively compact in \( X \). From the steps 1 and 2, follows that \( \Gamma(B) \) is relatively compact in \( C(I; X) \) and so that \( \Gamma \) is completely continuous. Finally, the theorem 1.1 assert that \( \Gamma \) has a fixed a in \( C(I; X) \). The proof is complete.

If the maps \( g, p, q \) fulfill some Lipschitz conditions instead of the compactness properties considered in the preceding theorem, we also can establish a result of existence.

**Theorem 3.2**

Assume that (H1) and (H4) are verified and that the following conditions hold:

(a) for every \( 0 < c < t < r > 0 \), the set \( U(t, t', r) = \{ (t', y) : y(t', 0) = 0 \} \) is relatively compact in \( X \).

(b) There exists positive constants \( l_0, l_1 \) such that,

\[
\| g(t, x_0) - g(t, x_1) \| \leq l_0 \| x_0 - x_1 \|, \quad \forall t \in I \times X
\]

\[
\| p(u) - p(v) \| \leq l_1 \| u - v \|, \quad \forall u, v \in C(I; X)
\]

\[
\| q(u) - q(v) \| \leq l_0 \| u - v \|, \quad \forall u, v \in C(I; X)
\]

and

\[
N(H S + l_0, a + \bar{N}(l_1 + l_0) + K N lim inf_{t \to \infty} W(t) = \varepsilon
\]

\[
\int_0^\infty m(s) ds + K_0 \sum_{i=1}^N (N P + \bar{N} Q) < 1
\]
Then there exists a mild solution of (1.1) – (1.5).

Proof

Let \( Y= \mathcal{C} (I;X) \) and \( \Gamma= \Gamma_1+\Gamma_2+\Gamma_3; Y\rightarrow Y \) be the map defined by

\[
\Gamma u(t)= \mathcal{C}(t)(x_0 + p(u(t)) + S(t)\psi(t) + q(u(t)) + g(0,x_{\omega(0)})) - \int_0^t \mathcal{C}(t-s)g(x_0,x_{\omega(0)})ds.
\]

We affirm that there exists \( r>0 \) such that \( \Gamma(B(0,Y)) \subset B(0,Y) \). In fact, let us assume the affirmation is false, then for each \( r>0 \) there exists \( \Gamma u \in \mathcal{C} (I;X) \) such that \( || \Gamma u || > r \), which imply that

\[
r \leq || \Gamma u || \leq N(|| H_0 || + || l_i || + || \varphi \varphi || + \sum_{n=1}^{\infty} \sum_{i=1}^{2n} \sum_{\alpha=1}^{2n} \frac{W_1(\xi)}{k} + \frac{m_j(\xi)}{k} +\sum_{n=1}^{\infty} (|| N || + || \mathbb{Q} || )
\]

which is an absurd.

Step 1

Let \( r>0 \) such that \( \Gamma(B(0,Y)) \subset B(0,Y) \) using the steps in the proof of theorem (3.1), follows that \( \Gamma_2 \) is completely continuous and from the estimate

\[
|| \Gamma_2 (x) - x || \leq (N(|| H_0 || + || l_i || + || \varphi \varphi || + \sum_{n=1}^{\infty} (|| N || + || \mathbb{Q} || )) || - 1 \leq \eta_0
\]

such that \( \Gamma_2 \) is a contraction.

Step 2

The map \( \Gamma_1 \) is a contraction on \( B(0,Y) \). The assertion follows directly from (3.1) and the estimate,

\[
|| \Gamma_1 (x) - x || \leq 2 (|| N || + || \mathbb{Q} || (|| - 1 \leq \eta_0
\]

Thus, \( \Gamma_1 \) is a condensing map on \( B(0,Y) \). The assertion is now consequence of the Sadovskii's point theorem, see [15,16].

The proof is finished.

Conclusion

In this section we consider the applications of our abstract result. We discuss the existence of solutions for the partial differential system with state-dependent delay and nonlocal conditions;

\[
\frac{\partial^2 u(t,x)}{\partial t^2} + \int_{0}^{T} a_b(t-s)u(s)ds - \int_{0}^{T} \int \rho(t,\theta)u(t,\theta)d\theta ds = \int_{0}^{T} f(t,s)ds + \int_{0}^{T} g(t,s)ds
\]

for \( t \in [0,a], x \in \mathbb{R} \), subject to the nonlocal conditions

\[
u(0,\xi) = x_0 + \sum_{n=1}^{\infty} Bn \mu(t,j), \quad \xi \in J
\]

and transform system (4.1) – (4.5) in to the abstract Cauchy problem (1.1) – (1.5). Moreover \( f \) is a continuous linear operator with \( || f(t)|| \leq L_1, || f(t) \varphi || \leq L_2, \) is continuous and \( \rho(t,\varphi) \leq \rho \) for every \( \varphi \in [0,a] \).
\[ d^2(t) = \left( \int_{-\infty}^{t} v(t,s) \, ds \right)^2 + d^2 \]

\[ d^2(t) = \left( \frac{\mu(t,s)^2}{\mu(s,s)} \right)^{1/2} \]

Case(i) Assume that \( \varphi \) satisfies (Remark 2.4). Then there exists a mild solution of (4.1) – (4.5).

Case(ii) If \( \varphi \) is continuous and bounded, then there exists a mild solution of (4.1) – (4.5)

References