On Statistically (I) - sequential spaces
Renukadeviz V* and Prakash B

Abstract
In this paper, we answer the question "Is the product of two statistically sequential spaces statistically sequential?" raised by Zhongbao and Fucai. By an example we see that products of two statistically sequential spaces need not be statistically sequential. Also, we give a necessary and sufficient condition for the product to be in most general case of I-sequential spaces. And we point out the error in proposition 2.2 of. Further, we prove that a subspace of a statistically sequential space need not be statistically sequential and we find a necessary and sufficient condition for a subspace to be in most general case of an I-sequential space. Finally, we develop the properties of the most general concept of I-sequential spaces.

Keywords
Phrases statistical convergence; I-convergent sequence; Sequential space

Introduction
The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [1,2] and Schoenberg [3]. If $K \subseteq \mathbb{N}$, then $K_n$ will denote the set $\{K: k \leq n\}$ and $|K_n|$ stands for the cardinality of $K$. The natural density of $K$ is defined by
\[
d(K) = \lim_{n \to \infty} \frac{|K_n|}{n},
\]
if the limit exists [4,5]. A sequence $(X_n)$ in a topological space $X$ is said to converge statistically [6] (or shortly $s$-converge) to $x \in X$, if for every neighborhood $U$ of $x$, $d(\{n \in \mathbb{N}: X_n \in U\}) = 1$. Any convergent sequence is statistically convergent but the converse is not true [7]. But in general, $s$-convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces. It has been discussed and developed by many authors [8-16].

The concept of $I$-convergence of real sequences [17,18] is a generalization of statistical convergence which is based on the structure of the ideal $I$ of subsets of the set of natural numbers. In the recent literature, several works on $I$-convergence including remarkable contributions by salat et al. have occurred [15-20]. The idea of $I$-Convergence has been extended from the real number space to a topological space [14] and to a normed linear space [21].

$I$-convergence coincides with the ordinary convergence if $I$ is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if $I$ is the ideal of subsets of $\mathbb{N}$ of natural density zero.

Throughout this paper, $(X, \tau)$ will stand for a topological space and $I$ for a nontrivial ideal of $\mathbb{N}$, the set of all positive integers. $X \rightarrow x$ denotes a sequence $(X_n)$ converging to $x$. Let $X$ be a space and $P \subseteq X$ a sequence $(X_n)$ converging to $x \in X$ is eventually in $P$ if $\{x_n\} \subseteq P$ for some $K \subseteq \mathbb{N}$; it is frequently in $P$ if $\{x_n\}$ is eventually in $P$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $P$ be a family of subsets of $X$. Then $\cup P$ and $\cap P$ denote the union $\cup \{P: P \in P\}$, and the intersection $\cap \{P: P \in P\}$, respectively. We recall the following definition [22].

Definition 1
If $X$ is a nonvoid set, then a family of sets is $I \subseteq 2^X$ is an ideal if (i) $A, B \in I$ implies $A \cup B \in I$ and (ii) $A \subseteq B, A \in I$, implies $B \in I$.

The ideal is called nontrivial if $I \neq \emptyset$ and $X \notin I$. A nontrivial ideal $I$ is called admissible if it contains all the singleton sets. Several examples of nontrivial admissible ideals may be seen in [17].

Definition 2
A sequence $(X_n)$ in $X$ is said to be $I$-convergent to $x \in X$ if for any nonvoid open set $U$ containing $x$, $\{n \in \mathbb{N}: X_n \notin U\} \notin I$ We call $x$ as the $I$-limit of the sequence $(X_n)$ [14].

Definition 3
Let $X$ be a space. $P \subseteq X$ is called a sequential neighborhood of $x \in X$, if each sequence convergent to $x \in X$ is eventually in $P$. $A \subseteq X$ is said to be $I$-sequentially closed if $A \subseteq X \setminus O$ has an $I$-limit in $O$ [1].

Definition 4
$O$ is $I$-sequentially open if and only if no sequence in $X \setminus O$ has an $I$-limit in $O$ [1].

Definition 5
A subset $A$ of a space $X$ is said to be $I$-sequentially closed if for every sequence $(X_n)$ in $A$ with $(X_n)$-$I$-converges to $x$, then $x \in A$ [1].

Definition 6
A topological space is $I$-sequential when any set $O$ is open if and only if it is $I$-sequentially open.

Definition 7
A space $X$ is called locally $I$-sequential if every point of $x \in X$ has a neighborhood which is an $I$-sequential space.

Even though we mainly deal with $I$-sequential spaces, we see the basic definition for $s$-sequential space since it will be useful for the examples which deal with $s$-sequential spaces. An $I$-sequential space $X$ is statistically sequential if $I=\{A\subseteq X: d(A)=0\}$.

Definition 8
A space $X$ is called statistically sequentially(or shortly s-sequential)
space [6] if for each non-closed subset $A \subset X$, there is a point $x \in X$ \ $A$ and a sequence $(x_n)_{n \in \mathbb{N}}$ in $A$ statistically converging to $x$.

There is another way to define $s$-sequential space.

Definition 9

A subset $A$ of a space $X$ is said to be a statistically sequentially open set ($s$-sequentially open) [25] if for any sequence $(X_n)$ statistically converges to $x$ and $x \in A$ then $\{|n: x_n \in A|\} = \infty$.

A topological space is $s$-sequential when any set $O$ is open if and only if it is $s$-sequentially open.

Definition 10

A subset $K$ of the set $\mathbb{N}$ is called statistically dense [6] if $d(k) = 1$.

Definition 11

A subsequence $S$ of the sequence $I$ is called statistically dense in $L$ [4] if the set of all indices of elements from $S$ is statistically dense.

Remark 12

1. The limit of an $I$-convergent sequence is uniquely determined in Hausdorff spaces [14,6].
2. If a sequence $(x_{n(i)})_{n(i)}$ converges to $x$ in the usual sense, then it statistically converges to $x$. But the converse is not true in general.
3. A sequence $(x_{n(i)})_{n(i)}$ is statistically convergent if and only if each of its statistically dense subsequence is statistically convergent.
4. If a sequence $(X_n)$ $I$-converges to $x$, then every subsequence $(X_{n(i)})_{n(i)}$ is $I$-convergent for every $I \in I$.

Lemma 13

Let $X$ be a topological space and $A \subset X$. Then the following hold [1].
(a) $A$ is $I$-sequentially open.
(b) $X \setminus A$ is $I$-sequentially closed.

$I$-sequential spaces 14

In this section, we answer the question 2 [25]: Is the product of two $s$-sequential spaces $s$-sequential?

By the Example 1 that product of two $s$-sequential spaces need not be $s$-sequential. Also, we give the necessary and sufficient condition for the product to be in most general case of $I$-sequential space. And we point out the error in proposition 2 of [1].

Example 1

For each $n \in \mathbb{N}$ let $S_n = \{x_{n,m} : m \in \mathbb{N}\}$ be a sequence statistically converges to $x$ such that $x_{n,m} \neq x_{n,k}$ if $m \neq k$. Take $x = \bigcup\{S_n : n \in \mathbb{N}\}$. Let $X'$ be the disjoint topological sum of $S_n$ where $n \in \mathbb{N}$ and $X$ be the space obtained from $X'$ by identifying $x_{n,m} \to x$. Now let $f : X' \to X$ be a natural mapping. Since each $S_n$ is $s$-sequential space and hence $X'$ by Proposition 1. By Theorem 1 in [25], $X$ is a $s$-sequential space.

$X = \bigcup\{S_n \setminus \{x_n\} : n \in \mathbb{N}\} \cup \{x\}$.

The open set in $X$ is as follows:
1. Each point $x_{n,m}$ is isolated;
2. Each open neighborhood of the point $x$ is a set $\mathcal{V}$ of the form $\mathcal{V} = \bigcup\{M_n : n \in \mathbb{N}\} \cup \{x\}$, where each $M_n$ is a dense subsequence of $S_n$. Next we define $Y$: Let $Y = \bigcup_{n=1}^{\infty}(S_n \setminus \{x_n\})$. Now let $Y = \bigcup_{n=1}^{\infty} \{y_n \times \{n\}\} \cup \{y\}$. Then $Y$ is a metric space.

Topologize $Y$ as follows:

Let each point of $\bigcup_{n=1}^{\infty} \{y_n \times \{n\}\}$ be open and $\{V_n(y)\}$ be a countably local base at $y$, where $V_n(y) = \bigcup_{n=1}^{\infty} \{y_n \times \{n\}\} \cup \{y\}$. Then $Y$ is a metric space.

Now let $A = \{(x_n, x_m) : x_n \neq x_m \in \mathbb{N}\}$. Thus, $A$ is not closed in $X \times Y$, and statistically sequentially closed in $X \times Y$ since $A$ has no non trivial $s$-convergent sequence. Suppose $A$ has a $s$-convergent sequence $(x_n(x,i))$ statistically converges to $(y)$, then $\pi(x_n(x,i))$ statistically converges to $x$ which implies for some $m$, $d([n \in \mathbb{N} / x_n(x,i)] = k > 0$. But the corresponding sequence in $X$, $\pi(x_n(x,i))$ statistically converges to $y$ which is a contradiction to $d([n \in \mathbb{N} / x_n(x,i)] \in V_{n,i}(y)) > 1 - k < 1$. Therefore, $X \times Y$ is not a $s$-sequential space.

Next we see the necessary and sufficient condition for the product of $I$-sequential spaces to be $I$-sequential.

Proposition 2

Every $I$-sequential space is a quotient of a topological sum of $I$-convergent sequences.

Proof. Let $X$ be an $I$-sequential space. For each $x \in X$ and for each sequence $(S_n)$ in $X$ converging to $x$, let $I(x, x) = \{s_j, j=1,2,3,\ldots\} \cup \{x\}$ be a topological space, where each $s_j$ is a discrete point and neighborhood $U$ of $x$ is such that $[n \in \mathbb{N} / x_n \neq x] = I$. Let $f(x, s) = x_{n, s}$ where $x$ be the set of all $I$-convergent sequences. Now we consider a mapping $f : X \to f(x, s) = x_{n, s}$.

1. $f$ is onto.

For each point $x \in X$ there is a constant sequence $S$ in $X$ such that $I(x, s) = \{s_j, j=1,2,3,\ldots\} \cup \{x\}$ that is, there exists $I(x, s) \subset X$ and $f(x, s) = x$. Therefore, $f$ is onto.

2. $f$ is continuous.

Let $U$ be an open set in $X$ and $(x', S) \in f^{-1}(U)$. Then there is a sequence $S$ in $X$ such that $x' = f(x, s) = x_{n, s}$ that is, $(x', S) \subset I(x, s)$ and $f((x', S)) = x'$. If $(x', S)$ is an isolated point, then there is nothing to prove. If $(x', S) = (x, S)$, then there exists $f \in I$. The following Example 1 shows that the products of two $s$-sequential spaces need not be $s$-sequential.
such that $S_n \subseteq U$ for $n \in \mathbb{N} \setminus I$ and hence $\{(x,S) : n \in \mathbb{N} \setminus I\} \subseteq f^{-1}(U)$ which is open in $I(S, X)$ and hence open in $X$. Therefore, $f^{-1}(U)$ is open in $X$. Therefore, $f$ is continuous.

3. $f$ is quotient.

Suppose $U \subseteq X$ and $f^{-1}(U)$ is open in $X$. If $x_n \in U$ and $S_n \subseteq S$ is $I$-convergent to $x_n$ in $X$, then $(x_n, S_n) \in f^{-1}(U) \cap I(S, X) (\pi(S_n))$ which is an open neighborhood of $(x_n, S_n)$ in $I(S, X) (S_n)$. Then as a subset of $I(S, X) (S_n)$, there exists $I \subseteq I$ such that $(S_n, S) \in f^{-1}(U) \cap I(S, X) (S_n) = f^{-1}(U)$ for $n \in \mathbb{N} \setminus I$, and hence $S_n \subseteq U$ for $n \in \mathbb{N} \setminus I$. Hence $U$ is $I$-sequentially open and thus open. Therefore, $f$ is a quotient mapping.

Here $I(S, X)$ is not second countable, when the set of all finite subsets of $\mathbb{N}$ is a proper subset to $I$ and not a Hausdorff space, when the set of all finite subsets of $\mathbb{N}$ properly contains $I$. Hence for such $I$, $I(S, X)$ is not a metrizable space and hence its topological sum.

In Proposition 2 in [1], $S = \oplus \bigcup_{n \in \mathbb{N}} X$ is mentioned as a metric space. But it is true only when $I$ is the set of all finite subset of $\mathbb{N}$.

Also, the following Proposition 3 is not true in general.

**Proposition 3**

Every $I$-sequential space $X$ is a quotient of some metric space.

Suppose Proposition 3 is true. Then by Proposition 2 [1], $X$ is an $I$-sequential space if it is a quotient image of a metric space. And hence $I$-sequential space and sequential space coincide, by Corollary 1.14 in [23] which is a contradiction to Example 2 in [25].

**Lemma 2.5**

Let $X$ be an $I$-sequential space. If $f : X \rightarrow Y$ preserves $I$-convergence sequence, then $f$ is continuous.

**Proof.** Let $U$ be an open subset in $Y$. Since $X$ is an $I$-sequential space, it is enough if we prove that $f^{-1}(U)$ is an $I$-sequentially open subset of $X$.

Let $\{X_n\}$ be a sequence in $X$ which $I$-converges to $x \in f^{-1}(U)$. By our assumption, $f(\{X_n\}) I$-converges to $y \in f(U)$ since $U$ is open in $Y$, for some $I \subseteq I$, $\exists f(x_n) \in U$ for $n \in \mathbb{N} \setminus I$. This implies $x_n \in f^{-1}(U)$ for $n \in \mathbb{N} \setminus I$, and hence $f^{-1}(U)$ is an $I$-sequentially open subset of $X$. Therefore, $f$ is continuous.

**Theorem 1**

Let $X$ and $Y$ be the spaces obtained from $X$, $Y$ as in Proposition 2. Then the following hold.

1. $X$ is an $I$-sequential space
2. $(X \times Y', \pi_1, \pi_2)$ is homeomorphic to $X \times Y'$
3. $X \times Y'$ is an $I$-sequential space.

**Proof 1**

Let $U$ be an $I$-sequential open subset of $X$ and $(s', S') \subseteq U$ where $S'$ is an $I$-convergent sequence $(x_n)_{n \in \mathbb{N} \setminus I} \subseteq S'$ with its limit $x$. Then $\{(x_n, S) : n \in \mathbb{N} \setminus I\} \subseteq f^{-1}(U)$ since $U$ is an $I$-sequentially open subset of $X$. This implies $\{(x_n, S) : n \in \mathbb{N} \setminus I\} \subseteq f^{-1}(U)$ since $U$ is an $I$-sequentially open subset of $X$. Therefore, $X$ is an $I$-sequential space.

**Proof 2**

Let $f : (X \times Y', \pi_1, \pi_2)$ be defined by $f((x, y, s), (y, \pi_1(s))) = (x, \pi(s), \pi_2(s))$. Since projection mappings $\pi_1, \pi_2$ are continuous and by Proposition 1 in [1], $\pi(s)$ and $\pi_2(s)$ are $I$-convergent sequences in $X$ and $Y$, respectively. Therefore, $f$ is well defined.

(a) $f$ is one-one and onto
Since product of $I$-convergent sequence is again $I$-convergent sequence, $f$ is also, and hence we easily check that $f$ is one-one.

(b) $f$ is continuous
Clearly, $f$ preserve $I$-convergence and by Theorem, $f$ is continuous.

(c) $f^{-1}$ is continuous
Let $U$ be an open set in $(X \times Y', \pi_1, \pi_2)$ and $(y', S') \subseteq (X \times Y', \pi_1, \pi_2)$. Then $f^{-1}(U)$ is an $I$-sequential open and thus open. Therefore, $(X \times Y', \pi_1, \pi_2)$ is homeomorphic to $X \times Y'$.

3. By (a), $(X \times Y')$ is an $I$-sequential space. By (b) and Proposition 2 in [1], $X \times Y'$ is an $I$-sequential space.

By Lemma 3 and Definition 6, we have the following Theorem 2.

**Theorem 2**

Let $X$ be a topological space. Then the following are equivalent:

(a) $X$ is an $I$-sequential space.

(b) Every $I$-sequentially open subset of $X$ is open.

(c) Every $I$-sequentially closed subset of $X$ is closed.

**Theorem 3**

Each $I$-sequentially open (closed) subspace of an $I$-sequential space is $I$-sequential.

**Proof.** Let $X$ be an $I$-sequential space. Suppose that $Y$ is an $I$-sequentially open subset of $X$. By Theorem 2, $Y$ is open in $X$. Let $U$ be an arbitrary $I$-sequentially open subset in $X$ and let $(x_n)_{n \in \mathbb{N} \setminus I}$ be a sequence in $X$ such that $x_n \in U$ for all $n \in \mathbb{N} \setminus I$. Since $U$ is an $I$-sequentially open subset of $X$, there exists $I \subseteq I$ such that $(x_n)_{n \in \mathbb{N} \setminus I} \subseteq U$. That is, $(x_n)_{n \in \mathbb{N} \setminus I} \subseteq U$. Therefore, $Y$ is an $I$-sequential space.

If $Y$ is an $I$-sequentially closed subset of $X$, then by Theorem 2, $Y$ is closed in $X$. Let $F$ be an $I$-sequentially closed subset in $Y$ and let $(x_n)_{n \in \mathbb{N} \setminus I}$ be a sequence in $F$ such that $x_n \in Y$ and hence $x_n \in F$. Therefore, $F$ is an $I$-sequentially closed subset in $X$ and hence it is closed in $X$. Since $Y$ is closed in $X$, $F$ is closed in $Y$. 

Volume 1 • Issue 1 • 1000103

Page 3 of 5

Proposition 4

Every locally I-sequential space X is I-sequential.

Proof. Let U be an I-sequentially open set in X. Since X is a locally I-sequential space, there exists an I-sequential neighborhood V of x. With out loss of generality, assume that V is open and an I-sequential space. Since U is I-sequentially open, U \cap V is an I-sequentially open subset of V. Since U is an I-sequential space, x \in U \cap V is open in V and hence in X which is contained in U. Therefore, U is open and hence X is an I-sequential space.

In the rest of the section, the space X and the mapping φ=f: X \times X are as in Proposition 2.3.

Theorem 4

The product of two I-sequential spaces X and Y is I-sequential if \phi_x, \phi_y is a quotient mapping.

Proof. Suppose \phi_x, \phi_y is a quotient mapping. By (c) in Theorem 1, X\times Y is I-sequential. By Proposition 2 in [1] and by assumption, X\times Y is I-sequential.

Conversely, suppose that X\times Y is I-sequential. Then \phi_x: (X\times Y)\to X is a quotient mapping and \phi_y \circ f: X\times Y\to Y\times Y is also a quotient mapping (definition of \phi_y: X\times Y\to Y\times Y). Thus, \phi_x \circ f^{-1} = \phi_x \circ \phi_y: X' \to Y' is a quotient mapping.

Since a s-sequential space is not closed under Cartesian product like a sequential space, naturally, one can arise a question that “Is subspace of a s-sequential space, s-sequential?” The answer is not as shown by the following Example 2.

Example 2

Let S be the s-convergent sequence with the limit a, for each n\in\mathbb{N} that is, S_n = \{x_n: m \in \mathbb{N}\} \cup \{a\} and \{x\} be a s-convergent sequence with its limit x. Let X be the topological sum of the sequence S, n=1,2,3,... and \{x\} \cup \{x\}. Since each S_n, n=1,2,3,... and \{x\} \cup \{x\} is s-sequential, X is s-sequential, by Proposition 1.

Now let Y be the space obtained from X by identifying a to x.

Let f: X \to Y be the natural mapping. By Theorem 4 in [25], Y is a s-sequential space but not a s-Fretchet Urysohn space.

Let W be a weak base consists of the following three types of collection of weak neighbourhood of x'

If x'=a then

T_{x'} = \{ \{x'\} \}

Basis of S_n, if x' \in S_n \backslash \{x\}

If x'=x, U \in T_x then implies that

U = \{ x, n \in \mathbb{N} \land d( x, n ) \} \cup \{ S_n \} \cup \{ x \} is a non-thin subsequence S_n, when n \in \mathbb{N}\cup\{x\}.

Then W is a weak base of Y. But for all U \in T_x, x \notin \text{Int}U Therefore, Y is not a s-Fretchet Urysohn space by Theorem 4 in [25].

Next we see the necessary and sufficient condition for which a subspace of an most general case I-sequential space is I-sequential.

Theorem 5

A subspace Y of an I-sequential space X is I-sequential if \phi_{x'}: \phi^{-1}_{x'}(U) is a quotient mapping.

Proof. Let Y' = \phi_{x'}(Y) and \phi' = \phi_{x'}|_{\phi^{-1}_{x'}(U)}, and let \phi_{x'}: Y' \to Y be a mapping as in the proof of Theorem 3. Suppose \phi' is a quotient mapping and let \phi_{x'}^{-1}(U) be open in Y where U \subset Y.

Since \phi' is quotient, it is enough if we prove that \phi^{-1}_{x'}(U) is open in Y'.

Let (s,S) \in \phi^{-1}_{x'}(U) where S = \{x, S\} \times \{x\}. Suppose (x,S) \in \phi^{-1}_{x'}(U) \cap \phi^{-1}_{x'}(V), then there is nothing to prove.

Suppose (s,S) \notin \phi^{-1}_{x'}(U) and (x,S) \notin \phi^{-1}_{x'}(V)

Let U' = \{(s,S): x \not\in \{s,S\}\}. Since (S(x),s), is open in X' and \phi_{x'}^{-1}(Y) is a subspace of X', I(s,S) \cap \phi_{x'}^{-1}(Y) = U' is open in \phi_{x'}^{-1}(Y).

Now let

Y_{n} = \left\{ x_{n}, \text{ if } x_{n} \in Y \right\}

\left\{ x, \text{ if } x \in \cap \phi_{x'}^{-1}(U) \right\}

Then \{Y_{n}\} \cap Y \text{ I-converges to } x \in U which implies (Y_{n}, S) \text{ I-converges to } (x,S) \in \phi_{x'}^{-1}(U).

That is

I = \{ n: (x_{n}, S) \not\in \phi_{x'}^{-1}(U) \}

\left\{ n: (x_{n}, S) \in \phi_{x'}^{-1}(U) \right\}
Proposition 5

The continuous open or closed image of a I-sequential space is I-sequential.

Proposition 6

If the product space is I-sequential, so is each of its factors.

References


Author Affiliation

Center for Research and Post Graduate Department of Mathematics, Ayya Nadar Janaki Ammal College, India